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Not all inequality measures were created equal

The measurement of wealth inequality, its decompositions, and an application to European household wealth

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Household Finance and Consumption Network (HFCN)

This paper contains research conducted within the Household Finance and Consumption Network (HFCN). The HFCN consists of survey specialists, statisticians and economists from the ECB, the national central banks of the Eurosystem and a number of national statistical institutes.

The HFCN is chaired by Ioannis Ganoulis (ECB) and Oreste Tristani (ECB). Michael Haliassos (Goethe University Frankfurt), Tullio Jappelli (University of Naples Federico II) and Arthur Kennickell act as external consultants, and Juha Honkkila (ECB) and Jiri Slacalek (ECB) as Secretaries.

The HFCN collects household-level data on households' finances and consumption in the euro area through a harmonised survey. The HFCN aims at studying in depth the micro-level structural information on euro area households' assets and liabilities. The objectives of the network are:

1) understanding economic behaviour of individual households, developments in aggregate variables and the interactions between the two;
2) evaluating the impact of shocks, policies and institutional changes on household portfolios and other variables;
3) understanding the implications of heterogeneity for aggregate variables;
4) estimating choices of different households and their reaction to economic shocks;
5) building and calibrating realistic economic models incorporating heterogeneous agents;
6) gaining insights into issues such as monetary policy transmission and financial stability.

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The paper is released in order to make the results of HFCN research generally available, in preliminary form, to encourage comments and suggestions prior to final publication. The views expressed in the paper are the author's own and do not necessarily reflect those of the ESCB.

Contents

Abstract

Much of the literature on inequality, both that on the theoretical features of inequality measurement and that on the discussion of the results of empirical analysis, has preferred to focus on income inequality. This paper looks into the analysis of wealth inequality, which can be performed by carefully adapting the techniques used in the case of income distributions. The paper focuses on the measurement of inequality itself and includes an application to European data on wealth. We summarise the main inequality measures used in the economic literature, expanding the focus to lesser known but relevant ones, grounding their use in socio-economic theory and highlighting the connections between them. In particular, we investigate how each measure captures the same movement in the wealth distribution and why different measures can lead to differences in the observed change in inequality over time or across countries. In the main theoretical contribution of the paper we obtain a novel decomposition of changes in inequality measures as a set of equalising and disequalising factors, which sheds some light on the different results across indicators. We complement the analysis by focusing on the decomposition of wealth inequality measures, gaining an understanding of the contributions of inequality by wealth component and socio-demographic characteristics. The distribution of wealth of European households obtained by the Household Finance and Consumption Survey (HFCS) in 2010 and 2014 is used for empirical analysis and application of our methods.

JEL codes: D31, D63

Keywords: wealth, inequality, Atkinson, Lorenz curve, Gini coefficient, Pietra index, decomposition

Non-technical summary

The growing interest in inequality, along with the widespread perception that inequality has been increasing in developed countries, merits a detailed discussion of how it is measured. In this vein, there are a number of considerations to take into account, namely i) the different kinds of inequality – wealth, income, well-being, multidimensional (a combination of these factors) or another concept; ii) whether an inequality analysis should be performed on a set of indicators or by means of a comparison of the distributions themselves; and iii) the specific indicators that could be used.

Much of the literature on inequality, both that on the theoretical features of inequality measurement and that on the discussion of the results of empirical analysis, has preferred to focus on income inequality. However, in this paper we are interested in the analysis of wealth inequality, which can be performed by carefully adapting the techniques used in the measurement of income inequality. As a practical application of our methods, we use the results of the second wave of the Household Finance and Consumption Survey (HFCS), published in December 2016, to understand the changes in the distribution of wealth in the euro area and to address the apparent issue that arises as a result: that inequality increases for most measures but decreases for some.

Our first building block focuses on the definition of inequality measures, which starts with the reasonable properties that such measures should have. We then present indicators from different branches of the literature: Atkinson's social welfare-based approach, which led to the **Atkinson index**; the **Generalised Entropy indices**, including **Theil's measure**; the **Lorenz curve**; the well-known **Gini coefficient** and the less well-known **Pietra index**. These different measures behave differently when there are negative values of wealth – which is not a concern in the analysis of income inequality – so we study which measures can be used for wealth distributions and under what conditions. We also show how the different families of measures are related, also touching briefly on the popular "top percent shares" and quantile ratios measures used in other strands of inequality research.

Our second building block concerns the measurement of changes in inequality and, specifically, Atkinson's absolute comparison criterion, which allows two distributions to be compared without reference to a specific inequality measure. We apply Atkinson's idea of an absolute criterion to the analysis of elementary transfers that always decrease inequality. This results in a novel decomposition of the change in inequality that takes a series of redistributive ("equalising") or anti-redistributive ("disequalising") moves. These two types of move lead to a unique decomposition of the changes in the Gini coefficient, which also carries over to the other inequality measures. This allows us to view a change between two distributions as the combination of transfers – some increasing the equality and others increasing the inequality. We tie this decomposition to the intersection of the Lorenz curves and show how the two are linked, with each segment between two intersection points corresponding to a strictly equalising or disequalising change.

Our third and final building block is to bring together the other decompositions of inequality measures proposed in the literature and highlight how they can be used based on the inequality measure and the type of decomposition (for example, by wealth component or by population sub-group). The Generalised Entropy indicators can be uniquely decomposed by population sub-groups into between and within components. The Gini coefficient can only be decomposed into these between and within components when the population sub-groups are ranked by wealth level and are non-overlapping on this variable; otherwise there will be a residual term. As regards wealth components, general results make it possible to decompose the inequality measures through marginal effects. From this we can see how the change in the value of a wealth component affects inequality.

These different building blocks are then applied to the results of the Household Finance and Consumption Survey, which collected granular information on the wealth of households in most euro area countries in 2010 and 2014. In the intervening period wealth inequality was found to have increased by a positive but modest amount for most inequality measures at the euro area level. There was, however, variation across countries and the discrepancies between measures were larger at the country level than at the euro area level. In these countries the presence of redistributive and anti-redistributive factors becomes apparent in our decomposition and we link the differences in the amount of change to the inequality measures' varying levels of sensitivity to the different parts of the distribution and to the presence of different sub-groups in the changes in the distribution of wealth. To investigate further, we estimate the sensitivity of the different measures to various wealth components and find that an increase in the value of the household's main residence or safe financial assets tends to reduce inequality, while an increase in the value of liabilities or self-employment businesses increases inequality.

1 Introduction

The purpose of inequality analysis is to compare the distribution of an underlying resource – in its changes over time, across countries or groups, or both – or to provide a measure of inequality of a distribution, with possibly a specific but not always explicit concept of "equality". The pursuit of both goals, following rigorous logical reasoning that starts from accepted requirements with socio-economic justification, has been described in a large literature; we refer to the seminal contributions of Dalton (1920) and Atkinson (1970) among many others, and point to the recently updated volume by Cowell (2011). It has nevertheless left many practitioners confused about the proper indicators to use, their properties and their interpretation, as evidenced by several papers that attempt to explain these elements (Jurkatis and Strehl, 2013; Kimhi, 2011).

The default choice in many papers on inequality has been the Gini coefficient. Although the Gini coefficient has many good properties, it does not lend itself well to proper decompositions over time or across groups and it may not be as sensitive to changes in the distribution as one could wish. A lot of work has also been carried out recently on the "top *x*% share" of income or wealth but the intrinsic properties of this indicator are rarely discussed. An extreme approach – and one we find unfortunate –is to reject an inequality measure either on (misplaced) theoretical grounds or because of practical difficulties in obtaining an accurate measurement.

We propose returning to the definition of the concept of inequality proposed by Atkinson (1970) and transposing it to a mathematical indicator or ranking criterion that will measure the level of inequality of a distribution or allow us to compare two distributions along the dimension of inequality. The level of inequality can be understood, on the one hand, as the distance to equality: in the case of a wealth distribution, measuring inequality is choosing a way of measuring the distance from the observed distribution to the distribution in the case of complete equality, i.e. the distribution when all households have the same wealth. On the other hand, comparing the inequality of two distributions may not always produce a clear answer, so we will determine the conditions under which it is possible to provide an answer. This is particularly the case when evaluating the changes in inequality over time, where different conclusions can be reached depending on the measure used; when this occurs it is important to determine why.

First it is necessary to review the main techniques for inequality analysis that have been proposed in the literature and study the similarities and differences between them. This will allow a better understanding of the varied results that may be obtained when measuring inequality using the different methods.

This investigation on the measurement of inequality requires the selected indicators, including Gini, Pietra, Atkinson, Theil and the class of Generalised Entropy indices, to be defined. These indicators can be linked to the Lorenz curve, which we will use to understand and compare distributions.

Of particular interest is the analysis of changes in inequality over time, which answers questions such as “why did inequality decrease?” or “who benefited the most from the change in the distribution of wealth?” One major difficulty is that the different inequality indicators will not show changes of the same sign or magnitude, and the answers to the questions will depend upon the specific indicator used. By returning to Atkinson’s (1970) absolute principle, it is possible to devise a novel technique for decomposing the changes over time into equalising and disequalising components, i.e. those that contribute to decreasing inequality and those that contribute to increasing it. Again, the components will be linked to the Lorenz curves of the two distributions by way of their intersection points. This approach clarifies the differing results that arise between indicators owing to their particular sensitivities to transfers along the wealth distribution. Using the inequality indicators it is also possible to apply decompositions and related techniques to assess the contributions to total inequality of population sub-groups or components that make up the analysed variable.

Our main testing ground for these measures will be the Household Finance and Consumption Survey (HFCS). The HFCS – an initiative of the ECB and the Eurosystem national central banks – provides data on the distribution of wealth of euro area households in two different periods, the first around 2010 and the second around 2014, and is therefore an excellent source for analysing recent trends in wealth inequality in a large set of European countries.

The results of the second wave of the survey, which were published in December 2016, reported a modest increase in wealth inequality in the euro area between 2010 and 2014. Several recent contributions (Lenza and Slacalek, 2018; Ampudia et al., 2018) either investigate the impact of monetary policy in general, and quantitative easing, in particular, on household inequality in the euro area, or investigate the drivers of wealth inequality in the short and in the long term (e.g. Lieberknecht and Vermeulen, 2018). Our approach here is different, and clearly in the camp of measurement theory, as we focus on the measurement of inequality in itself, and not on the possible driving factors behind it. A careful application of several techniques introduced in the first chapters of this paper will be used to investigate the changes in equality between 2010 and 2014 and highlight the different situations across countries.

The paper is organised as follows: Chapter 2 is devoted to the problem of measuring inequality. It starts by listing the basic requirements that an inequality measure should satisfy then explains the processes for obtaining different indicators, giving definitions and providing indications for interpretation. Other measurement attempts, relationships between the measures, and procedures for dealing with negative values of the variable are also discussed. Chapter 3 presents different techniques for measuring changes in inequality over time and introduces the main theoretical development of the paper: a method to decompose the differences between two distributions into equalising and disequalising components. Chapter 4 presents and discusses different decompositions of the inequality indicators and related techniques for assessing the contributions to inequality of population sub-groups or wealth components. These include the property of additive decomposability, the computations of marginal effects for some indicators, and the decomposition of the change in time of the Gini coefficient. The relation between intersections of Lorenz curves and decompositions by population sub-groups is also investigated. Finally, in Chapter 5 we illustrate some of the techniques of inequality analysis by using HFCS data on the wealth distribution of euro area households. In particular, we investigate the reported modest increase in wealth inequality in the euro area between 2010 and 2014 by inspecting the results in each country and using different inequality analysis techniques.

2 The measurement of inequality

Several measures of inequality have been introduced in the literature, mostly in the context of income distributions. This section presents several indicators and describes their use as tools for measuring the inequality of wealth distributions.

In this section, we first list the four properties that an inequality indicator should have. Then, we describe three different approaches for obtaining inequality measures: i) using social welfare notions; ii) looking for a convenient mathematical structure; and iii) building on the Lorenz curve. These lead to the definitions of Atkinson (1970), Generalised Entropy (including the Theil index, 1967), Gini (1914) and Pietra (1915) indicators. In the third part we introduce other widely used measures in the analysis of inequality, including the “top shares”, or shares of wealth owned by specific groups at the top of the distribution, and the ratios of percentiles. In the fourth part we present the relationships between all the inequality measures, before detailing how to deal with negative and zero values of the distribution for each of the inequality measures.

2.1 Main properties of inequality measures

The main properties that appear in the literature as basic requirements for an inequality indicator are as follows.[1]

Anonymity: the indicator remains unchanged if the individuals of the distribution are permuted. In other words, all individuals are interchangeable and all that matters is their wealth, not their name.

Population principle (Dalton, 1920): the indicator remains unchanged if the distribution is replicated a finite number of times.

Principle of transfers (Pigou, 1912; Dalton, 1920): if a transfer of value $t > 0$ is made from an individual with wealth y_i to another with wealth y_j such that $y_i - t > y_j + t$, the indicator decreases. In other words, a transfer that takes from the rich and gives to the poor, but the rich remain richer than the poor, can only cause inequality to decrease. This transfer is known as a Pigou-Dalton transfer. This principle could be termed “strict”, while the “relaxed” version only requires that the indicator not increase in the presence of a positive Pigou-Dalton transfer.

Scale invariance: the indicator remains unchanged if the distribution is scaled by a constant factor. In other words, there is no monetary illusion in the measurement of inequality.[2]

These four properties ensure that the indicators behave in a reasonable manner. In particular, scale invariance means that the units used to measure wealth can be

[1] Definitions adapted from Cowell (1998) and Shorrocks (1980).

[2] This can be contrasted with the principle of additive invariance, which is used in some of the literature.

disregarded. However, these properties are not sufficient to single out a measure or a family of measures, and additional properties may be assumed in order to fix the uniqueness of the measure (see Section 2.2.2). Finally, indicators that do not necessarily respect these principles, or that respect other principles, have been proposed. We will briefly touch upon these in the sections below.

2.2 Indicators selected to measure inequality

Different approaches can be followed when selecting specific indices from among those displaying the properties introduced above. We detail three approaches in the following subsections. First, we present the welfare approach described by Atkinson (1970), which is based on the use of a social welfare function and leads to the definition of the family of eponymous inequality measures. Second, we consider the requirement that the indicator possess the property of decomposability. This approach results in the definition of the Generalised Entropy family of indices, which includes Theil's entropy measure. Finally, we investigate the Lorenz curve of the distribution, which is a natural way of obtaining the Gini and Pietra indicators.

2.2.1 Atkinson's measures and the social welfare approach

Let F be a distribution of wealth. The equally distributed equivalent level of wealth, $\xi(F)$ is the level of wealth per household which, if equally distributed, would give the same level of social welfare as the distribution F. It is such that $W\big(\xi(F)\big) = W(F)$ where W is a social welfare function given by $W = \int u(y)dF(y)$, where u is a utility function. Let $\mu(F)$ denote the mean of distribution F.

The **Atkinson index** (Atkinson, 1970) is, in the most general sense, defined as the normalised ratio of the equally distributed equivalent level of wealth to the mean of the actual wealth distribution.

$$\text{Atk} = 1 - \frac{\xi(F)}{\mu(F)}$$

It depends on the choice of the social welfare function. To ensure that the properties introduced in Section 2.1 are displayed, the choice of the form of the utility function is limited. The family of Atkinson indices referred to in the literature is obtained by taking the utility function $u(y) = \frac{y^{1-\varepsilon}-1}{1-\varepsilon}$ when $\varepsilon \neq 1$ and $u(y) = \ln y$ when $\varepsilon = 1$. The parameter $\varepsilon > 0$ is interpreted as accounting for the level of aversion to inequality.

In the case of an empirical distribution with n elements, where y_i denotes the wealth of household i and $\bar{y}$ the sample average, the Atkinson indices can be expressed as:

$$\text{Atk}(\varepsilon) = 1 - \frac{1}{\bar{y}}\left(\frac{1}{n}\sum_{i=1}^{n} y_i^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}} \text{ if } \varepsilon \neq 1$$

$$\text{Atk}(1) = 1 - \frac{1}{\bar{y}}\left(\prod_{i=1}^{n} y_i\right)^{\frac{1}{n}}$$

The Atkinson index lies between zero and one and increases with inequality. It is equal to zero in the case of complete equality, i.e. when all individuals have the same (positive) wealth, and tends to one in the case of complete inequality, i.e. when one individual has all the wealth and all others have nothing at all.

The parameter $\varepsilon > 0$ accounts for the level of aversion to inequality in the social welfare function. A low ε implies a low degree of inequality-aversion. As ε rises, more weight is attached to transfers at the lower end of the distribution and less weight to transfers at the top and thus the level of inequality-aversion increases.

In general, Atkinson indices can only be computed for positive values of the variable. For some values of $\varepsilon > 0$, they can also be computed for zero and negative values of the variable.

The value of the Atkinson index can be interpreted in terms of the equally distributed equivalent described above. For example, an Atkinson index equal to 0.7 means that, if wealth was equally distributed, the same level of social welfare could be achieved with only 30% of the actual total wealth.

The indicators that make up the family of Atkinson indices respect the four properties presented in Section 2.1: anonymity, the population principle, the principle of transfers, and scale invariance.

2.2.2 Theil's measure and the Generalised Entropy indices

An inequality indicator is said to display the property of additive decomposability, defined by Shorrocks (1984), if it can be decomposed by population sub-groups and expressed as a weighted sum of a within-group and a between-group component.

The **Generalised Entropy (GE)** measures constitute the only family of indicators (up to a transformation) that display additive decomposability as well as anonymity, the population principle, the principle of transfers, and scale invariance[3]. The GE measures depend on a parameter α that expresses the sensitivity of the indicator to different parts of the distribution. The special cases of $\alpha = 1$ and $\alpha = 0$ are known as the **Theil index** and the **mean log deviation**, respectively. The case of $\alpha = 2$ is equal to half the squared coefficient of variation.

In the case of an empirical distribution with n elements where y_i denotes the wealth of household i and $\bar{y}$ the sample average, the Generalised Entropy indices can be expressed as:

[3] More precisely, a continuous inequality measure respecting anonymity and the population principle satisfies the principle of transfers, scale invariance and decomposability if and only if it is ordinally equivalent to a Generalised Entropy indicator for some parameter alpha. See Cowell (2000) for details.

Indicator	Formula
Generalised Entropy (α ∉ {0,1})	$\mathrm{GE}(\alpha) = \frac{1}{\alpha(\alpha-1)}\frac{1}{n}\sum_{i=1}^{n}\left(\left(\frac{y_i}{\bar{y}}\right)^{\alpha} - 1\right)$
Theil (GE(α = 1))	$\mathrm{GE}(1) = \frac{1}{n}\sum_{i=1}^{n}\frac{y_i}{\bar{y}}\ln\left(\frac{y_i}{\bar{y}}\right)$
Mean log deviation (GE(α = 0))	$\mathrm{GE}(0) = -\frac{1}{n}\sum_{i=1}^{n}\ln\left(\frac{y_i}{\bar{y}}\right)$

The Generalised Entropy indicators are equal to zero in the case of complete equality, i.e. when all individuals have the same wealth. A larger value of the index indicates larger inequality in the distribution. The GE is unbounded when considered for a theoretical distribution but is bounded in the case of a finite independent and identically distributed sample (for example, in the case of the Theil index, the bound is $\ln n$, where n is the sample size, and it corresponds to the case of complete inequality).

The Generalised Entropy indices can be computed for all values of the variable when $\alpha \neq 1$ is a positive integer. They can only be computed for non-negative values of the variable in certain cases, depending on the parameter α chosen.

The special case of Theil's measure (GE$(\alpha = 1)$) was introduced by Theil (1967) as a consequence of Shannon's information theory. Theil's measure is given by the difference between the maximum possible entropy and the observed entropy of the wealth distribution. Indeed, the entropy of the distribution of net wealth shares, where the vector $z = (z_1, z_2, \ldots, z_n)$ represents the wealth share of all observed households $z_i \coloneqq yi/\sum_{i=1}^{n} y_i$ and where each household has the same weight, is given by

$$H(z) = \sum_{i=1}^{n} z_i \ln\left(\frac{1}{z_i}\right)$$

The maximum possible entropy corresponds to complete equality in the distribution, i.e. where all households have the same (positive) wealth. In this case $z_i = \frac{1}{n}\ \forall i$ and thus $H(z) = \ln n$. In the opposite case, complete inequality, i.e. when one household has all the wealth and all others have nothing at all, we have $\exists i\ s.t.\, z_i = 1$ and $z_j = 0\ \forall j \neq i$ and thus $H(z) \underset{z \to 0}{\longrightarrow} 0$.

Theil's measure is, then, the difference between the maximum possible entropy and the observed entropy of the system:

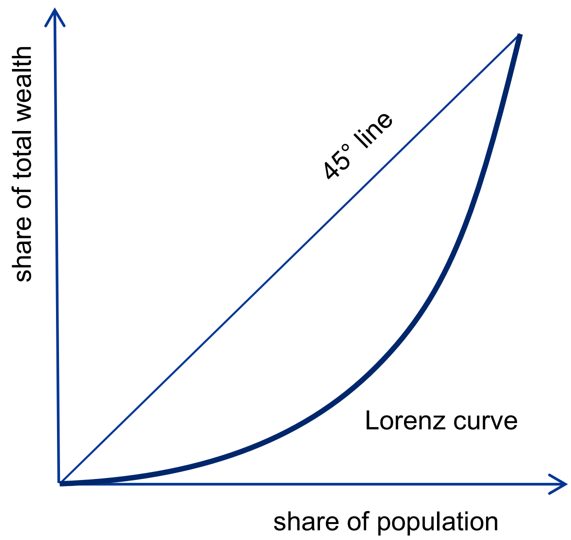
$$\text{Theil} = \mathrm{GE}(\alpha = 1) = \sum_{i=1}^{n} z_i \ln(nz_i) = \frac{1}{n}\sum_{i=1}^{n}\frac{y_i}{\bar{y}}\ln\left(\frac{y_i}{\bar{y}}\right)$$

2.2.3 Gini's and Pietra's measures and Lorenz curves

The **Lorenz curve** of a distribution of wealth plots the proportion of total wealth belonging to the bottom $x\%$ of the population. The Lorenz curve is increasing, convex and lies in the first quadrant of the Cartesian plane in the case of positive values of the variable. It is equal to the 45-degree line in the case of complete equality, i.e. it represents the distribution that would arise if all the households had the same wealth. In all the other cases it lies below the 45-degree line. Intuitively, the closer the Lorenz curve is to the 45-degree line, the more equal the distribution is.

More formally, in the case of a distribution of wealth F, for $0 \le q \le 1$ the quantile functional is defined as $Q(F,q) \coloneqq \inf\{y|F(y) \ge q\} \coloneqq y_q$ and the cumulative wealth functional is defined as $C(F,q) \coloneqq \int_{-\infty}^{Q(F,q)} y\, dF(y)$. The Lorenz curve of the distribution is the normalised cumulative wealth functional and plots the quantiles q against $L(F,q)$ where $L(F,q) = C(F,q)/\mu(F)$ and $\mu(F)$ is the mean of the distribution F.

Figure 1
Lorenz curve of a distribution



The Lorenz curve gives an intuition for the definition of both the Gini indicator and the Pietra indicator. The **Gini coefficient** (Gini, 1912) corresponds to the normalised area between the Lorenz curve of the distribution and the 45-degree line:

$$G = 1 - 2\int_0^1 L(F,q)dq$$

The Gini coefficient can be expressed in several equivalent forms, such as the normalised average absolute difference between all pairs of wealth in the population or the covariance formula: $G = \frac{2}{\bar{y}} cov\big(y, F(y)\big)$.

In the case of an empirical distribution with n elements, y_i denotes the wealth of household $i \in \{1, \dots, n\}$, the vector $y = (y_1, y_2, \dots, y_n)$ represents the wealth of all observed households and $\bar{y}$ is the sample average. In the case where y is such that $y_1 \le y_2 \le \cdots \le y_n$, we define $X_i = \frac{i}{n}$ as the cumulative share of population and

$Y_i = \frac{\sum_{j=1}^{i} y_j}{\sum_{j=1}^{n} y_j}$ as the cumulative share of wealth. The Gini coefficient can then be expressed as:

$$G = 1 - \sum_{i=0}^{n-1} (X_{i+1} - X_i)(Y_{i+1} + Y_i) = \frac{2\sum_{i=1}^{n} i\, y_i}{n \sum_{i=1}^{n} y_i} - \frac{n+1}{n}$$

The **Pietra index** (Pietra, 1915), also known as the Ricci index (Ricci, 1916), the Schutz index or the Hoover index, is calculated as the mean relative absolute deviation to the mean, or as the maximum vertical distance between the Lorenz curve of the distribution and the 45-degree line.

The Pietra index has a simple interpretation; it is the proportion of total wealth that would have to be redistributed in order to achieve complete equality of the wealth distribution, i.e. when all individuals have the same wealth.

In the case of an empirical distribution with n elements where y_i denotes the wealth of household i and $\bar{y}$ the sample average, the Pietra index can be expressed as:

$$P = \frac{1}{2n} \sum_{i=1}^{n} \frac{|y_i - \bar{y}|}{\bar{y}}$$

Both the Gini and the Pietra indices are equal to zero in the case of complete equality, i.e. when all individuals have the same (positive) wealth. When calculated for non-negative values of the variable, both indicators tend to one in the case of complete inequality, i.e. when one individual has all the wealth and all others have nothing at all. The Gini and Pietra measures can be calculated for all values of the variable but in that case are not bounded above. A larger value of the measures indicates larger inequality in the distribution.

The Gini index respects anonymity, the population principle, the principle of transfers and scale invariance. The Pietra index respects anonymity, the population principle and scale invariance but it does not respect the (strict) Pigou-Dalton principle of transfers. This can easily be seen by taking the definition of the indicator as the maximum vertical distance between the Lorenz curve and the 45-degree line. A Pigou-Dalton transfer can be made by modifying only one part of the Lorenz curve, which does not affect the point of the maximum vertical distance. This reduces the area between the Lorenz curve and the 45-degree line and, consequently, the value of the Gini coefficient diminishes without the Pietra index being affected. The Pietra index would, however, respect a weaker version of the Pigou-Dalton principle of transfers, that is, a non-strict decrease.

2.3 Other measures

The indicators presented in the previous sections, namely the Atkinson and Generalised Entropy families and the Gini coefficient, have the main properties that, according to the literature, a useful inequality measure should possess (the Pietra index does not respect the Pigou-Dalton principle of transfers in the strict sense).

There are, however, other indicators that are widely used in the analysis of wealth inequality; the most popular include the shares of wealth owned by specific groups of households, such as the wealthiest 5%, and the ratios of quantiles of the distribution.

The **top wealth share** indicators (popularised by Piketty and co-authors mainly for income distributions) are defined as the share of total wealth owned by the households that are above the corresponding percentile of the wealth distribution. For example, the top 10% share gives the proportion of total wealth owned by the households that lie to the right of the 90th percentile. Top shares are bounded below by the share itself and would be bounded above by 100% in the case of non-negative values of net wealth. A higher value of the top share indicator indicates higher concentration of wealth.

The **ratios of quantiles** are defined as the ratio of the corresponding percentiles of the distribution. A P90/P50 value of four (4) means that the household at the 90th percentile of the distribution is four times wealthier than the household at median wealth. In the case of a distribution of wealth, the lower percentiles can be negative or very close to zero. This directly affects the denominator of the percentile ratios and undermines the interpretation of the value of the indicator. The ratios of quantiles are bounded below by 1 (in the case of complete equality, i.e. when all the households have the same wealth, all quantile ratios are equal to 1). Higher values of the quantile ratios indicate a wider gap in net wealth quantiles.

The top wealth shares and quantile ratio indicators respect anonymity, the population principle and scale invariance but they do not respect the (strict) Pigou-Dalton principle of transfers. For this reason, they are seldom considered as inequality measures in the literature. For example, the top 10% indicator equal to 50% means that half of total wealth is owned by the 90% poorest households and the other half by the 10% wealthiest but no information is given on the distribution of wealth within each of the two groups. If half of the wealth was equally distributed among the 90% poorest or if the household at the 90th percentile alone owned half of total wealth, the value of the top 10% indicator would remain the same.

2.4 Relationships between the measures

The Generalised Entropy and Atkinson indicators were presented independently in the previous sections but they are in fact formally related. The Atkinson indicators can be transformed into GE indicators via a strictly increasing transformation, given by the following expression:

$$\mathrm{GE}(1-\varepsilon) = \frac{\left(1-\mathrm{Atk}(\varepsilon)\right)^{1-\varepsilon}-1}{\varepsilon(\varepsilon-1)}$$

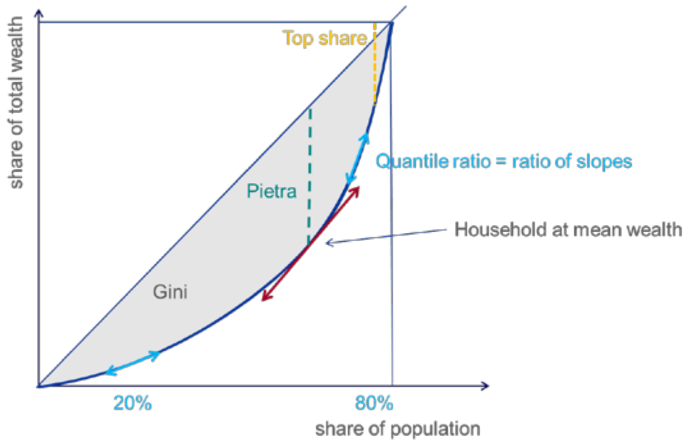
This relationship implies that the Atkinson index with parameter $\varepsilon > 0$ and the corresponding GE index with parameter $\alpha = 1 - \varepsilon < 1$ induce the same ranking of distributions.[4] However, this transformation does not preserve the additive

[4] In particular, $\mathrm{GE}(0) = -\ln\left(1\text{-}\mathrm{Atk}(1)\right)$.

decomposability property of the GE indices. The Atkinson indices do, however, respect a weaker property of decomposability (Shorrocks, 1984).

Figure 2

Relationships between indicators and Lorenz curve



The other measures presented in this paper (Gini, Pietra, top shares, ratios of percentiles) can all be linked to the Lorenz curve of the distribution.

The Gini corresponds to the normalised area between the Lorenz curve of the distribution and the 45-degree line, $\text{Gini} = 1 - 2\int_0^1 L(F,q)dq$.

The Pietra index gives the maximum vertical distance between the Lorenz curve and the 45-degree line, $\text{Pietra} = \max_q\{q - L(F,q)\}$, with the distance obtained at the point in the Lorenz curve that corresponds to the household with mean wealth.

The share of wealth owned by the top x% is given by the distance from the Lorenz curve at point $q = 1 - x$ to the horizontal line at ordinate = 1, top $x\% = 1 - L\big(F, q(1-x)\big)$.

The ratio of percentiles, p_1/p_2, is the ratio of slopes of tangents to the Lorenz curve at points p_1, p_2, $\frac{p_1}{p_2} = \frac{\mathrm{d}L(p_1)}{\mathrm{d}q} / \frac{\mathrm{d}L(p_2)}{\mathrm{d}q}$ when the derivatives exist.

Besides the relationship with the Lorenz curve, the Gini coefficient can be shown to relate to the average of top shares, $\bar{t}$; as the size of the population goes to infinity, the Gini converges to $2\bar{t} - 1$.

2.5 Computations in the case of negative values

Some of the Generalised Entropy and Atkinson indices are not properly defined for negative and zero values of the variable. The Lorenz curve of the distribution is also affected by the presence of negative and zero values of the variable. Indeed, in this case, as the lower percentiles of the distribution are negative or zero, the Lorenz curve will lie below zero and will be strictly decreasing for the first percentiles of the wealth distribution.

Nevertheless, the Gini coefficient and the other measures related to the Lorenz curve can be computed taking into account all values of the variable[5]. They will be affected in some way, for example in the boundaries of the possible values assumed by the indicators. For ratios of percentiles with a denominator that is negative or very close to zero, the value of the ratio is severely affected. The shares of total wealth owned by the bottom x% of the distribution will also be negative or zero in certain cases.

Different ad hoc procedures can be applied when relevant for analysis, such as deleting observations with non-positive values, replacing non-positive values with €1 or replacing negative values with €0. These are, however, outside the scope of this paper. The detailed restrictions on the computations along with the possible procedures applied, depending on the choice of parameter, are provided in the tables below for the Generalised Entropy and the Atkinson indicators.

Table 1
Generalised Entropy – variable restrictions

Parameter	Restriction on the variable
$\alpha \in \mathbb{N}, \alpha \neq 1, \alpha \neq 0$	No restriction, $x \in \mathbb{R}$
$\alpha = 0$ or $\alpha = 1$	$x > 0$
$-\alpha \in \mathbb{N}$	$x \neq 0$
$\alpha \in \mathbb{Q}\backslash\mathbb{Z}, \alpha > 0$	$x \geq 0$
$\alpha \in \mathbb{Q}\backslash\mathbb{Z}, \alpha < 0$	$x > 0$
In other cases, $\alpha \in \mathbb{R}\backslash\mathbb{Q}$	$x > 0$

Table 2
Atkinson – variable restrictions

Parameter	Restriction on the variable
$\varepsilon = 1$	$x > 0$
$\varepsilon \in \mathbb{N}, \varepsilon \neq 1$	$x \neq 0$
$\varepsilon \in \mathbb{Q}\backslash\mathbb{N}, 0 < \varepsilon < 1$	$x \geq 0$
$\varepsilon \in \mathbb{Q}\backslash\mathbb{N}, \varepsilon > 1$	$x > 0$

These procedures for dealing with non-positive values of net wealth have different effects on the overall distribution. Taking the example of the second wave of the HFCS, we can see that net wealth is negative for around 5% of euro area households, with important differences across countries (the percentages range from 1% in Malta to 14% in the Netherlands in 2014)[6]. The value of the indicator as a measure of the inequality of the entire distribution is thus affected differently in each country. However, no important differences arise in the measurement of the change in time of

[5] The only exceptions are the ratios of percentiles that cannot be computed when the percentile in the denominator of the ratio is equal to zero.

[6] According to the second wave of the Household Finance and Consumption Survey.

the indicator provided that the share of households with negative or zero wealth is similar in both cases.

3 The measurement of changes in inequality

The previous section discussed different measures of the inequality of a distribution and, more specifically, a wealth distribution. However, the analysis of wealth inequality does not only cover the levels of inequality of a distribution, as given by the value of one of these indicators, it also involves a comparison of several wealth distributions. When looking at comparisons of distributions referring to the same population at different moments in time, an analysis of the changes can provide interesting results.

In the first part of this section we introduce the criterion established by Atkinson to compare two distributions according to inequality using the Lorenz curves of the distributions. In the second part we propose a method for analysing the changes in the distribution by only analysing Pigou-Dalton transfers between convenient transformations of the analysed distributions. The different impact each transfer has on each specific inequality indicator is then assessed.

3.1 Atkinson's comparison criterion

Instead of using a measure of inequality like the indicators presented above, it would be useful to have a way of comparing two given distributions without having to assign a numeric value to each considered distribution. The comparison criterion proposed by Atkinson (1970) follows a reasoning similar to the deduction of the eponymous indices presented in Section 2.2.1. The indicators depend on a social welfare function relying on a specific form of a utility function. In the case where a wider class of utility functions is considered, Atkinson shows that two distributions can be compared by considering their Lorenz curves, provided that these curves do not intersect.

This **comparison criterion is absolute** in the sense that it does not depend on the social welfare utility function (provided that it is increasing and concave).

The ranking of distributions according to the **Atkinson criterion** is always possible if the Lorenz curves of the distributions do not intersect. Three situations can occur: i) the Lorenz curve of distribution A always lies above the Lorenz curve of distribution B, meaning distribution A is more equal than distribution B; ii) the Lorenz curve of distribution A always lies below the Lorenz curve of distribution B, meaning distribution A is more unequal than distribution B; or iii) the Lorenz curves of distributions A and B intersect, meaning it is not possible to rank them according to this absolute notion of inequality.

Atkinson further interprets the result, stating that two distributions can be compared independently of the utility function (other than being increasing and concave) if and only if one distribution can be obtained from the other by redistributing wealth from the richer to the poor, that is, exclusively by means of Pigou-Dalton transfers. This also implies that two distributions can be compared by Atkinson's Lorenz curve criterion if and only if **all the inequality indicators adhering to the principle of transfers rank them in the same way**. These results will be further developed in the next sections.

3.2 Decomposing changes into Pigou-Dalton transfers

The inequality measures differ in their sensitivity to the shape of the wealth distribution and these differences may lead to opposite conclusions being drawn in practice on the change in wealth inequality over time. Indeed, the latest results of the Household Finance and Consumption Survey show that, in some countries, one indicator points towards an increase in wealth inequality while another suggests a decrease.

To understand the origins of these differences, we express the total change between any two wealth distributions as a finite sequence of transfers between pairs of elements of the distribution, such that each transfer is either a Pigou-Dalton transfer or a "reverse", or "anti-" Pigou-Dalton transfer, that is, either a redistributive (or equalising) or an anti-redistributive (or disequalising) factor. These transfers always lead to an unequivocal change in inequality in all inequality measures that conform to the principle of transfers described above. We will then see how each indicator reacts differently to the same transfer, depending on its amount, the ranks of the two elements, and their wealth.

3.2.1 Measuring the contribution from a Pigou-Dalton transfer

A Pigou-Dalton transfer is defined as the positive wealth transfer from the richer to the poorer of two individuals such that the richer individual is still richer than the poorer individual after the transfer has occurred. If distribution y^{i+1} is obtained from y^i through a Pigou-Dalton transfer, we say that y^i is obtained from y^{i+1} through a "reverse" or anti-Pigou-Dalton transfer. A Pigou-Dalton transfer always decreases inequality according to the (strict) Pigou-Dalton principle of transfers (see Section 2.1) and will thus cause the inequality indicator to decrease (or remain unchanged, when the non-strict Pigou-Dalton principle of transfers is respected). Similarly, an anti-Pigou-Dalton transfer will cause the indicator to increase or remain unchanged.

Let $c_t^y(y_k, y_j)$ denote the transfer of amount t performed on distribution[7] y from individual y_k to individual y_j. $c_t^y(y_k, y_j)$ denotes a Pigou-Dalton transfer when $y_j < y_j + t < y_k - t < y_k$ and an anti-Pigou-Dalton transfer when $y_j + t > y_j > y_k > y_k - t$. Total wealth is unchanged.

Let y^{i+1} be obtained from y^i by a Pigou-Dalton transfer $c_t^i(y_k, y_j)$, $j < k$. The effects of this transfer on the different inequality indicators introduced in the previous sections are as follows[8]:

$$\Delta\text{Gini}(c_t^i) \coloneqq \text{Gini}(y^{i+1}) - \text{Gini}(y^i) = \frac{2t(j-k)}{n\sum y_i} = \frac{2}{n}\frac{t}{\bar{y}}\frac{j-k}{n}$$

7 The individuals y_k are sorted ascendingly, i.e., $y_k \leq y_{k+1}\ \forall k$.

8 We note the transfer $c_t^i = c_t^i(y_k, y_j)$ for a lighter reading.

$$\Delta\text{GE}(0)(c_t^i) \coloneqq \text{GE}(0)(y^{i+1}) - \text{GE}(0)(y^i) = -\frac{1}{n}\left[\ln\left(\frac{y_j+t}{y_j}\right) + \ln\left(\frac{y_k-t}{y_k}\right)\right]$$

$$\Delta\text{GE}(1)(c_t^i) \coloneqq \text{GE}(1)(y^{i+1}) - \text{GE}(1)(y^i)$$
$$= \frac{1}{n}\left[\frac{y_j}{\bar{y}}\ln\left(\frac{y_j+t}{y_j}\right) + \frac{y_k}{\bar{y}}\ln\left(\frac{y_k-t}{y_k}\right) + \frac{t}{\bar{y}}\ln\left(\frac{y_j+t}{y_k-t}\right)\right]$$

$$\Delta\text{GE}(\alpha)(c_t^i) \coloneqq \text{GE}(\alpha)(y^{i+1}) - \text{GE}(\alpha)(y^i)$$
$$= \frac{1}{\alpha(\alpha-1)}\frac{1}{n}\left[\left(\frac{y_j+t}{\bar{y}}\right)^\alpha - \left(\frac{y_j}{\bar{y}}\right)^\alpha + \left(\frac{y_k-t}{\bar{y}}\right)^\alpha - \left(\frac{y_k}{\bar{y}}\right)^\alpha\right]$$

$$\Delta\text{Pietra}(c_t^i) \coloneqq \text{Pietra}(y^{i+1}) - \text{Pietra}(y^i)$$
$$= \frac{1}{2n}\left[\frac{|y_j+t-\bar{y}|}{\bar{y}} - \frac{|y_j-\bar{y}|}{\bar{y}} + \frac{|y_k-t-\bar{y}|}{\bar{y}} - \frac{|y_k-\bar{y}|}{\bar{y}}\right]$$

For transfers of a fixed amount t, the impact on the Gini coefficient only depends on the distance between the relative positions of the individuals in the distribution, not on their absolute wealth. The Gini attaches more weight to transfers between elements of the distribution whose rankings are more distant, irrespective of their wealth or their absolute rank.

Again for transfers of a fixed amount t, the Generalised Entropy with parameter $\alpha = 0$ attaches more weight to transfers at the bottom of the distribution, that is, when the amount of the transfer t is large when compared to the wealth of the individuals. The effects on GE(1) depend on a combination of factors. The ratio of the wealth of each individual concerned to the mean plays a role and, the larger the ratio, the larger the impact on the GE if α is large.

The impact on the Pietra index depends on the wealth of the individuals, before and after the transfer, in relation to the mean. If the wealth of all individuals concerned is smaller or larger than the mean, the Pietra index remains unchanged. In the other cases, three situations can occur for a Pigou-Dalton transfer, producing an impact on the Pietra index of:

$$\Delta\text{Pietra}(c_t^i) = \begin{cases} \frac{y_j-\bar{y}}{n\bar{y}} \text{ if } y_j < \bar{y} < y_j+t < y_k-t < y_k, \\ -\frac{t}{n\bar{y}} \text{ if } y_j < y_j+t < \bar{y} < y_k-t < y_k, \\ \frac{\bar{y}-y_k}{n\bar{y}} \text{ if } y_j < y_j+t < y_k-t < \bar{y} < y_k, \\ 0, \text{ in all other cases.} \end{cases}$$

3.2.2 Obtaining a sequence of Pigou-Dalton transfers

We now present an algorithm with M steps for obtaining a sequence of distributions starting from the initial distribution y^1 and ending in the final distribution y^M such that distribution y^{i+1} can be obtained from distribution y^i in the previous step by a Pigou-Dalton or an anti-Pigou-Dalton transfer.

First, the two distributions being compared are transformed so that they both have the same mean and number of elements. These transformations do not affect the inequality of the distributions because our definition of inequality respects scale invariance and the population principle. Second, the distributions are sorted in ascending order by their wealth, in line with the property of anonymity.

Each step of the algorithm[9] consists of the following actions:

- the difference between the current distribution and the final distributions is computed, giving the sorted list of individuals who need to increase their wealth and those that need to decrease it;
- the first individual who needs to decrease their wealth gives the most they can give (either all they can give or what the other needs to receive) to the first individual who needs to increase their wealth;
- the elements of the distribution are then sorted.

Applying the algorithm starting from distribution y^1 until distribution y^M is reached will result in a sequence of distributions S.

The total change in the inequality indicator can then be expressed as a sum of the contributions to the change from each step. The set of all the Pigou-Dalton transfers in S is denoted $PD(S)$ and the set of all the anti-Pigou-Dalton transfers is denoted $aPD(S)$. In the case of the Gini coefficient the expressions are as follows:

$$\text{Gini}(y^M) - \text{Gini}(y^1) = \sum_{i=1}^{M-1} \Delta\text{Gini}\left(c_t^i\right) = \sum_{c_t^i \in PD(S)} \Delta\text{Gini}\left(c_t^i\right) + \sum_{c_t^i \in aPD(S)} \Delta\text{Gini}\left(c_t^i\right)$$
$$= C_{PD}^{\text{Gini}} + C_{aPD}^{\text{Gini}}$$

The total contributions to the change in the Gini from Pigou-Dalton and anti-Pigou-Dalton transfers are $C_{PD}^{\text{Gini}} \leq 0$ and $C_{aPD}^{\text{Gini}} \geq 0$, respectively. Each of the contributions can be equal to zero when there are either no Pigou-Dalton transfers or no anti-Pigou-Dalton transfers in the sequence. In this case, the two distributions can be compared according to Atkinson's comparison criterion introduced in Section 3.1 and one of the distributions will be more equal than the other.

More generally, for any inequality indicator I, there exists a sequence S of distributions transforming y^1 into y^M by Pigou-Dalton and anti-Pigou-Dalton transfers such that the change in the indicator can be decomposed as follows:

$$I(y^M) - I(y^1) = C_{PD}^I + C_{aPD}^I$$

As seen in Section 3.2.1, the same Pigou-Dalton (similarly, anti-Pigou-Dalton) transfer will be measured differently by each inequality indicator, resulting in different values for the total contribution from Pigou-Dalton transfers, C_{PD}^I. Consequently, the total change in inequality will be measured differently by each of the indicators, reflecting

[9] The algorithm is detailed further in Appendix section entitled "Decomposition of inequality changes by Pigou-Dalton transfers".

the indicators' sensitivity to the distribution and resulting in changes in opposite directions in certain cases.

Theorem (minimum Pigou-Dalton sequence): the sequence of Pigou-Dalton and anti-Pigou-Dalton transfers following from the algorithm minimises the absolute value of the total contribution to the Gini coefficient from Pigou-Dalton transfers (C_{PD}^{Gini}).

Proof: see the section entitled "Decomposition of inequality changes by Pigou-Dalton transfers" in the Appendix. ■

The theorem ensures that the Pigou-Dalton sequence, although not necessarily unique, provides a unique decomposition of the total change in the Gini in two terms.

4 Decompositions of inequality measures

Having presented different inequality measures (in Chapter 2) and shown how the changes between distributions can be decomposed (in Chapter 3), in this chapter we look at how decomposition techniques can be useful for investigating the contributions to total inequality of different sub-groups of the population or of components that add up to the analysed variable. The population can be partitioned into relevant sub-groups according to, for example, age groups, net wealth quintiles, number of household members or countries. In the analysis of net wealth, the disaggregation of the variable into different types of real and financial assets can be performed.

Not all the inequality indicators make it possible to obtain decompositions with satisfactory socio-economic interpretations as it is often unclear how the decomposition should account for different changes in the population sub-groups or variable components. Another main issue is to deal with terms that are not unambiguously assigned to a specific sub-group or component and are then qualified as residuals or interaction terms. Moreover, in some cases different decompositions may be deduced for the same inequality indicator, without a strong justification for using one instead of another. Nevertheless, one can find decompositions and related techniques that are useful in practice.

This section is organised as follows: after a brief first section on notation, in Section 4.2 we provide a decomposition of the Generalised Entropy indicators by population sub-groups, followed in Section 4.3 by different decompositions of the Gini coefficient by population sub-groups. In Section 4.4 we propose a useful and novel way of partitioning the total population into sub-groups for analysing the changes in inequality, motivated by the relationship between the Gini decomposition by non-overlapping population sub-groups and the intersections of Lorenz curves. Section 4.5 deals with the assessment of the contributions of the components adding up to total net wealth. In the case of the Gini coefficient, three different techniques are introduced: decomposition by variable components, the elasticity of the wealth components in terms of marginal effects, and a decomposition of the changes of the indicator over time. We finish in Section 4.6 with a discussion on a systematic approach, initiated by Shorrocks, for obtaining decompositions by variable components for a class of inequality indicators, complemented with the elasticities of other inequality indicators.

4.1 Notation

Let y_i represent the net wealth of household $i \in \{1, \dots, n\}$. $\bar{y}$ is the sample average.

Population sub-groups: let X represent the complete population formed by n households that can be partitioned into K sub-groups of n_k households each, such that $X = \cup_{k=1}^{K} X_k$ and $n = \sum_{k=1}^{K} n_k$. $\overline{y_k}$ is the sample average of sub-group X_k.

Variable components: the variable net wealth can be expressed as a sum of R components such that $y_i = \sum_{r=1}^{R} y_{ir}$ where y_{ir} is the value of component r for household i. $\bar{y_r}$ is the sample average of component r. F_r denotes the theoretical distribution of the wealth component r.

In the theoretical sections that follow, we take the case in which all households have the same weight $\frac{1}{n}$. Small adjustments in the computations are needed to take into account different household weights; these are presented in Appendix section entitled “Expressions for decompositions of inequality indicators”.

4.2 Decomposition of Generalised Entropy indicators by population sub-groups

The Generalised Entropy indicators have the property of additive decomposability as introduced in Section 2.2.2. Therefore they can be decomposed by population sub-groups and expressed as a weighted sum of a within-group and a between-group component. The within component accounts for the inequality that exists inside each group whereas the between component accounts for the inequality across the groups. The Generalised Entropy computed over the entire population, $\mathrm{GE}_\alpha(X)$, can then be expressed as:

$$\mathrm{GE}_\alpha(X) = \underbrace{\sum_{k=1}^{K} \frac{n_k}{n} \left(\frac{\bar{y}_k}{\bar{y}}\right)^{\alpha} \mathrm{GE}_\alpha(X_k)}_{\text{within component}} + \underbrace{\mathrm{GE}_\alpha(\cup_{k=1}^{K} X_k)}_{\text{between component}}$$

where $\mathrm{GE}_\alpha(X_k)$ is the value of the GE indicator computed for the households belonging to sub-group X_k:

$$\mathrm{GE}_\alpha(X_k) = \frac{1}{\alpha(\alpha-1)} \frac{1}{n_k} \sum_{i=1}^{n_k} \left(\left(\frac{y_i}{\bar{y}_k}\right)^{\alpha} - 1\right)$$

and $\mathrm{GE}_\alpha(\cup_{k=1}^{K} X_k)$ is the between-group component, given by:

$$\mathrm{GE}_\alpha(\cup_{k=1}^{K} X_k) = \frac{1}{\alpha(\alpha-1)} \sum_{k=1}^{K} \frac{n_k}{n} \left(\left(\frac{\bar{y}_k}{\bar{y}}\right)^{\alpha} - 1\right)$$

The within component is calculated as the weighted sum of the value of the indicator in each of the K sub-groups. The between-group component is calculated as the value of the indicator of a distribution with K elements, each having as net wealth the mean of net wealth in the corresponding group and as weight the population share of the corresponding group.

4.3 Decomposition of the Gini coefficient by population sub-groups

Unlike the Generalised Entropy measures, the Gini coefficient cannot, in general, be decomposed by population sub-groups as a sum of a within and a between component. However, a decomposition into two components, within and between, does arise when the population is partitioned into "non-overlapping" groups in terms of wealth, i.e. when $\min_{y_i}(X_k) > \max_{y_i}(X_{k-1})\ \forall k$. The expression (adapted from Dagum (1997) and Alvaredo (2011)) is as follows:

$$\text{Gini} = \text{within} + \text{between} = \underbrace{\sum_{k=1}^{K} G_k \frac{n_k}{n}\frac{n_k\overline{y_k}}{n\bar{y}}}_{\text{within component}} + \underbrace{\sum_{k=1}^{K}\sum_{h=1}^{k-1} G_{kh}\left(\frac{n_k}{n}\frac{n_h\overline{y_h}}{n\bar{y}} + \frac{n_h}{n}\frac{n_k\overline{y_k}}{n\bar{y}}\right)}_{\text{between component}}$$

The within component is the weighted sum of the Gini coefficients of the sub-groups, G_k, and the between component is the weighted sum of the factors accounting for the relationships between each pair of sub-groups, G_{kh}, taking $k > h$ and sorted sub-groups.

$$G_k = \frac{\sum_{i=1}^{n_k}\sum_{j=1}^{n_k}\left|y_{ik} - y_{jk}\right|}{2n_k^2\overline{y_k}}$$

$$G_{kh} = \frac{\sum_{i=1}^{n_k}\sum_{j=1}^{n_h}\left|y_{ik} - y_{jh}\right|}{n_k n_h(\overline{y_k} + \overline{y_h})} = \frac{\overline{y_k} - \overline{y_h}}{\overline{y_k} + \overline{y_h}}$$

In general, in any population sub-group decomposition, a term that can be qualified as a residual will arise. It is argued to account mostly for interactions between the sub-groups but its allocation is ambiguous. Different decompositions of the Gini indicator by population sub-groups have been proposed and discussed in the literature, including decompositions by Bhattacharya and Mahalanobis (1967) or Dagum (1997). The decomposition proposed by Dagum writes the Gini as a sum of three factors, as follows:

$$G = \underbrace{G_w}_{within} + \underbrace{G_{nb} + G_t}_{between}$$

Dagum (1997) gives a socio-economic interpretation to each of the three factors in terms of their contributions to total inequality.[10] The term G_t is equal to zero in the case of non-overlapping population sub-groups and, in that case, this decomposition is equivalent to the decomposition into two terms presented at the beginning of this subsection.

Another decomposition of the Gini coefficient by population sub-groups has been proposed by Podder (1993). It is similar to the decomposition by variable components of Lerman and Yitzhaki (1985) and expresses the Gini as a sum of K components corresponding to the K sub-groups that add up to the complete population:

[10] The details are given in Dagum (1997). The three factors can be named as follows: G_w, within component; G_{nb}, net contribution of the extended Gini inequality between sub-populations; G_t, contribution of the income intensity of transvariation between sub-populations.

$$G = \sum_{k=1}^{K} \frac{n_k \overline{y_k}}{n\bar{y}} C_k$$

where C_k is the concentration coefficient of the k-th population sub-group vector $\underline{x_k}$. The elements of the sub-group are $x_{ik} = y_i \mathbb{1}_{\{i \in X_k\}}$ and the concentration coefficient is computed as the Gini coefficient of the curve that plots the cumulative proportions of vector $\underline{x_k}$ against the cumulative proportions of the total population ordered in ascending order according to their wealth. The concentration coefficient provides an indication of the relative position of the wealth distribution of the specific sub-group when compared to the wealth distribution of the complete population.

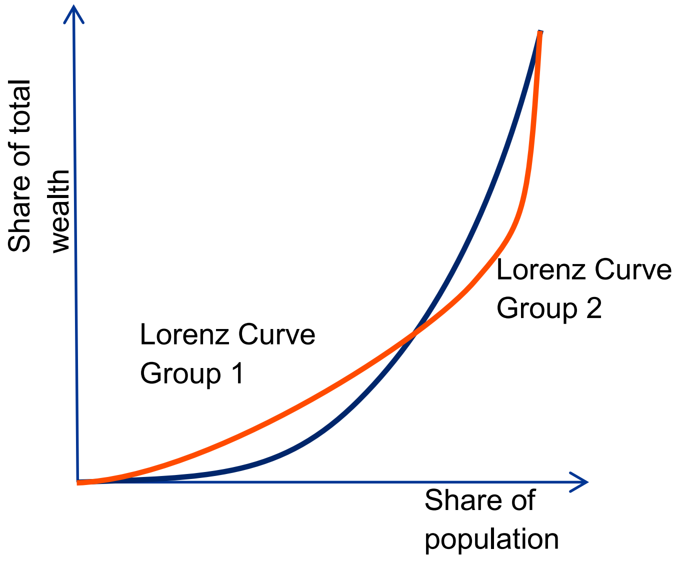
4.4 Non-overlapping population sub-groups given by the intersection of Lorenz curves and their link to the Gini index decomposition

According to Atkinson's comparison criterion introduced in Section 3.1, distribution y^1 can be considered more unequal (respectively, equal) than distribution y^2 if and only if the Lorenz curve of y^1 always lies below (respectively, above) the Lorenz curve of distribution y^2 (excluding of course the points (0,0) and (1,1)).

In all other cases, the Lorenz curves of the distribution intersect at least once. These intersections define non-overlapping population sub-groups and the intersection points correspond to the individuals for which the cumulative difference between the distributions is zero. We name these **Lorenz curve groups (LCG)**.

Figure 3

Lorenz curve groups



Indeed, the Lorenz curve of the distribution y^1 can be written as $LC_i^1 = \frac{\sum_{j=1}^{i} y_j^1}{\sum_{j=1}^{n} y_j^1}$.

Without loss of generality, we consider distributions y^1 and y^2 to be sorted, to have the same means and the same number of elements. We write the difference between two distributions $d = (d_1, \dots, d_n)$ where $d_i = x_i^1 - x_i^2 \; \forall i \in \{1, \dots, n\}$ and define the cumulative difference as $D_i = \sum_{j=1}^{i} d_j$.

It follows that the two Lorenz curves intersect when the cumulative difference D_i is equal to zero:

$$LC_i^1 = LC_i^2 \Leftrightarrow \frac{\sum_{j=1}^{i} x_j^1}{\sum_{j=1}^{n} x_j^1} = \frac{\sum_{j=1}^{i} x_j^2}{\sum_{j=1}^{n} x_j^2} \Leftrightarrow \sum_{j=1}^{i} \left(x_j^1 - x_j^2\right) = 0 \Leftrightarrow \sum_{j=1}^{i} d_j = 0 \Leftrightarrow D_i = 0$$

The LCG sub-groups – defined by taking all the individuals in between those for which the cumulative difference is equal to zero – have the same total wealth and the same number of elements and thus the same means. As seen in the expressions introduced in the previous sections, the between components of the decompositions by population sub-groups of both the Gini and the Generalised Entropy indicators only depend on the means and number of elements of the sub-groups; therefore, they will remain unchanged when transforming y^1 into y^2 for LCG sub-groups. This implies that distribution y^2 is obtained from y^1 via transfers occurring exclusively inside each of the LCG sub-groups. Moreover, each sub-group can be taken separately, producing a situation in which Atkinson's comparison criterion allows for a comparison of the distributions restricted to the sub-group. We conclude that, when transforming y^1 into y^2, wealth transfers only occur inside each of the LCG sub-groups and that, in each of these sub-groups, either exclusively Pigou-Dalton or anti-Pigou-Dalton transfers take place[11].

4.5 Contributions of wealth components: the case of the Gini coefficient

4.5.1 Decomposition by wealth components

Taking the covariance formula for the Gini coefficient, $G = \frac{2}{\bar{y}} \mathrm{cov}\big(y, F(y)\big)$, the decomposition by variable components (Lerman and Yitzhaki, 1985) is expressed as:

$$G = \sum_{r=1}^{R} C_r S_r = \sum_{r=1}^{R} P_r G_r S_r = \sum_{r=1}^{R} \underbrace{\frac{\mathrm{cov}(y_r, F)}{\mathrm{cov}(y_r, F_r)}}_{P_r} \underbrace{\frac{2\mathrm{cov}(y_r, F_r)}{\bar{y}_r}}_{G_r} \underbrace{\frac{\bar{y}_r}{\bar{y}}}_{S_r}$$

The two terms C_r and S_r correspond to the concentration coefficient (or "pseudo-Gini") and the share of total wealth of component r, respectively. The concentration coefficient is the product of the "Gini correlation" between wealth component r and total wealth, P_r, and the real Gini of component r, G_r.

11 We note nonetheless that, in practice, the intersection points of Lorenz curves may not correspond precisely to an individual of the distribution.

C_r, the concentration coefficient of a wealth component r, also known as the "pseudo-Gini", gives an indication of the relation of the distribution of wealth component r alone when compared to the distribution of total wealth. It is defined as one minus twice the area under the concentration curve, i.e. it is calculated in the same way as the Gini but with reference to the concentration curve instead of the Lorenz curve. The concentration curve plots the cumulative proportions of the wealth component r against the cumulative proportions of the population organised in ascending order according to their total net wealth. It can thus lie above the 45-degree line, for example when a wealth component is owned mostly by the households at the bottom of the distribution. The concentration coefficient is then bounded between -1 and 1, in the case of non-negative net wealth.

This decomposition is a valid mathematical identity but its interpretability can be questioned, as will be discussed in Section 4.6. Nevertheless, the expression of this decomposition will be shown to be the most useful for obtaining results in the following section.

4.5.2 Marginal effects of wealth components

The decomposition of the Gini by wealth components presented in the previous section can be useful in obtaining the elasticity of the indicator, which leads to the analysis of marginal effects, i.e. investigating how small changes in specific net wealth components will affect total net wealth inequality. The Gini elasticity with respect to the mean of component r measures the percentage change that would arise in total inequality as measured by the Gini coefficient if component r was uniformly increased by 1%. It was derived by Lerman and Yitzhaki (1985) following the decomposition by variable components presented in Section 4.5.1:

$$\eta_r(G) = \frac{S_r(C_r - G)}{G}$$

The sign of the elasticity of a component indicates whether it has an equalising or disequalising effect. Indeed, if $\eta_r(G) > 0 \Leftrightarrow C_r - G > 0$, component r has a disequalising effect in the distribution.

4.5.3 Contributions of wealth components to the changes in time

Following from the expressions of the Gini decomposition by variable components and the marginal effects presented in the precedent sections, a procedure for analysing the changes in time in the Gini coefficient can be obtained. The changes in time are then decomposed into two factors: i) the change in inequality within each component that makes up net wealth ("change in inequality"); and ii) the change in inequality that comes from a change in the allocation of the different components, i.e. the change in the share of each component in total net wealth ("change in share").

Redistribution of wealth within one component (e.g. financial assets) will affect the change in inequality factor but keep the relative shares unchanged, while a proportional increase of the value of a component will keep all within-component inequalities constant but will affect the shares. A disproportionate increase of business assets for the wealthy households, for example, will affect both factors and thus combine to increase inequality as a whole.

The expression has been proposed by Jurkatis and Strehl (2013)

$$\dot{G} = \underbrace{\sum_r (C_r - G)\,\dot{S}_r}_{\text{change in share}} + \underbrace{\sum_r S_r \dot{C}_r}_{\text{change in inequality}}$$

The expression can be adapted for computing the changes between the time periods $t-1$ and t as follows:

$$\Delta G_t = \underbrace{\sum_r \left(\frac{C_{r,t} + C_{r,t-1}}{2} - \frac{\left(G_{r,t} + G_{r,t-1}\right)}{2}\right) \Delta S_{r,t}}_{\text{change in share}} + \underbrace{\sum_r \frac{S_{r,t} + S_{r,t-1}}{2} \Delta C_{r,t}}_{\text{change in inequality}}$$

4.6 Shorrocks' systematic approach to decompositions by variable components

Shorrocks (1982) applied a systematic approach to the problem of obtaining decompositions by variable components for most inequality measures[12]. The analysis includes all the indicators that can be expressed as $\sum_i a_i y_i$, where a_i is the coefficient corresponding to a specific inequality measure. Gini, Pietra, Generalised Entropy and Atkinson are therefore included.

The goal is to find a decomposition that satisfies the condition "one source, one term", i.e. to find a decomposition that can be written as $\sum_r F_r$ where F_r is the contribution of the wealth component r. The main technical issue is to decide on the component to which terms arising owing to interactions between several components should be allocated. There are many ways to perform this allocation, giving rise to distinct decomposition expressions. To narrow down the possible decomposition expressions following a logical set of socio-economic assumptions, Shorrocks proposes a list of requirements that the decomposition should meet. These would give a theoretical justification for the choice of factors by allocating the interaction term. One of these requirements is, however, controversial, as it states that the contribution of a component to total wealth inequality is zero if this component is equally distributed among all households. This contradicts the fact that the addition of the same positive quantity to all the elements of the distribution decreases inequality, which is a property displayed by all the inequality indicators that obey to the principle of transfers and scale invariance. This critique has lead Paul (2004) to propose an assumption of negativity, which states that the contribution of a component to total wealth inequality is negative if this component is equally distributed among all households.

[12] The original reasoning by Shorrocks refers to decompositions by income components. The same applies to decompositions by wealth components.

From a combination of requirements set by both Shorrocks and Paul, decomposition rules can be obtained for some indicators but there is no consensus in the literature on the proper use of these decompositions. Still, the expressions of the decompositions are valid mathematical identities that can be used to derive the elasticities of the indicators, which have clear interpretations and are useful in practice. Other than the case of the Gini coefficient, already presented in Section 4.5.2, the elasticities for the Pietra and Generalised Entropy indices are derived as follows, adapting the reasoning of Paul:

$$\eta_r(GE(0)) = \frac{(S_r - \overline{S_r})}{GE(0)}$$

$$\eta_r\big(GE(\alpha)\big) = \frac{\alpha S_r\big(GE(\alpha)'_r - GE(\alpha)\big)}{GE(\alpha)}, 0 < \alpha$$

$$\eta_r(P) = \frac{S_r(P'_r - P)}{P}$$

With the “pseudo-indicators” defined as:

$$GE(1)'_r = \frac{1}{n}\sum_{i=1}^{n}\frac{y_{ir}}{\overline{y_r}}\ln\left(\frac{y_i}{\overline{y}}\right)$$

$$GE(\alpha)'_r = \frac{1}{\alpha(\alpha-1)}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{y_{ir}}{\overline{y_r}}\left(\frac{y_i}{\overline{y}}\right)^{\alpha-1} - 1\right), 0 < \alpha < 1$$

$$P'_r = \frac{1}{n}\sum_{i,y_i>\overline{y}}\frac{y_{ir} - \overline{y_r}}{\overline{y_r}}$$

5 Applications to European household wealth

We now apply some of the techniques of wealth inequality analysis introduced above using the distribution of the net wealth of European households as measured by the Household Finance and Consumption Survey.

The first section of this chapter describes the main features of the HFCS, a harmonised cross-country survey providing household-level microdata that are comparable across countries and thus able to be used for the analysis of the wealth distribution in the euro area in the first half of the 2010s. In the second section we present some stylised facts about wealth in the euro area. Third, using the HFCS we investigate the modest increase in wealth inequality in the euro area between 2010 and 2014 shown in the results of the second wave of the survey, which were published in December 2016. Several inequality measures are used in the analysis and country differences are also assessed. In the fourth section we use the decomposition of the total change in the distribution into Pigou-Dalton and anti-Pigou-Dalton transfers, presented in Section 3.2, to further investigate the different country results by indicator. In the fifth section, we use intersections of Lorenz curves to perform sub-group analysis and understand the different inequality trends across countries. In the sixth and final section, we analyse the contributions of the components making up net wealth to overall inequality and its change in time.

5.1 The Household Finance and Consumption Survey

The HFCS is a harmonised cross-country survey that collects household-level data on households' wealth, income and consumption along with related economic and demographic variables. The micro-dataset provides insights into the economic behaviour and financial situation of households.

The HFCS is conducted by the Household Finance and Consumption Network (HFCN), which is composed of researchers, statisticians and survey specialists from the ECB, European national central banks, some national statistical institutes and a number of experts in the field of household finances.

The survey is conducted every three years and the results for the first two waves were published in 2013 and 2016, with fieldwork mainly carried out in 2010/11 and 2013/14, respectively[13]. In the first wave, more than 62,000 households from 15 euro area countries were surveyed. All the 18 euro area countries (at the time) plus Hungary and Poland participated in the second wave, with over 84,000 households being interviewed.

[13] See the *Methodological report for the second wave of the HFCS* (2016) for country details.

The HFCS is a good source of data for estimating the wealth distribution of euro area households. Since wealth distribution is highly uneven, with certain financial instruments being held almost exclusively (and in large quantities) by the wealthiest households, the sampling method has been designed accordingly. Indeed, most of the countries participating in the HFCS oversample the wealthy via different methods[14]. This partially addresses the fact that wealthier households tend to be more difficult to contact and less likely to respond. In addition, the most important variables (including net wealth) have been subject to multiple imputation to correct item non-response, i.e. plausible values based on the information collected from other households were assigned to a variable when it was not collected or not collected correctly.

The resulting micro-dataset used for the computations in this paper is weighted at the household level and multiply imputed. The weights take into account the unit's probability of selection, coverage issues, unit non-response and an adjustment of weights to external data. The sum of the weights is equal to the total number of households in the population. The HFCN also provides replicate weights to be used in the estimation of the variance following the Rao-Wu rescaled bootstrap method.

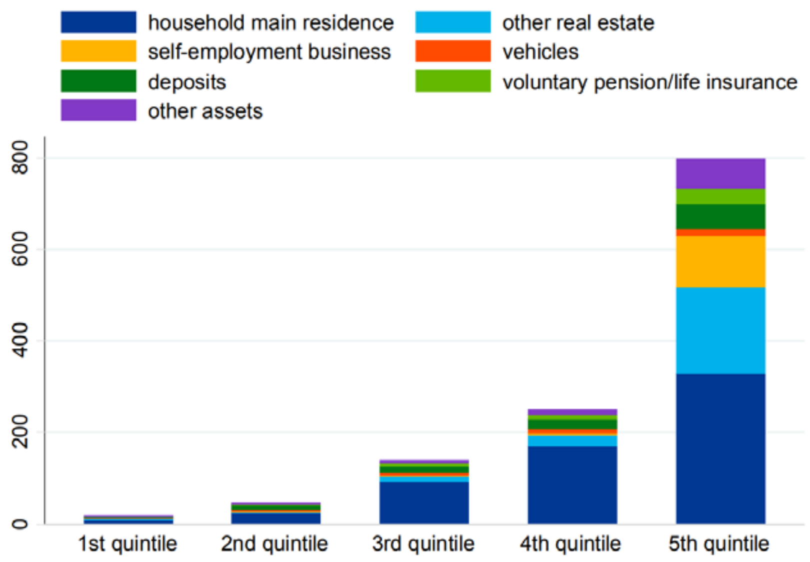
5.2 Household wealth in the euro area: general facts

This section briefly describes some of the main features of the distribution of assets, liabilities and net wealth in the euro area, as available in the HFCS. Interested readers are invited to refer to the HFCS reports on the results from the first and second wave (HFCN 2013 and 2016b).

Chart 1

Average portfolio by net wealth quintile, euro area 2014

(EUR thousands)



Source: HFCS, reproduced from HFCN (2016b).
Note: Hungary and Poland are not included.

[14] Idem.

Regarding concentration of assets, 48% of all assets are owned by the wealthiest 10% of the population. Moreover, the composition of wealth strongly varies by wealth level, with the poorest households having their wealth mostly in vehicles and valuables, the median households mostly in their main residence, and only the wealthiest households having a sizeable share of their wealth in their own businesses.

Average wealth levels vary by country (see in particular Box 4.1 of HFCN, 2016b), but the heterogeneity of households' wealth within a country dwarfs the heterogeneity between countries.

In terms of inequality, Table 3 shows the different measures described in Section 2.

Table 3
Wealth inequality indicators

	2010 wave		2014 wave	
	Measure	Standard error	Measure	Standard error
Inequality measures				
Atkinson index ($\varepsilon = 1$)	0.708	0.006	0.713	0.005
Atkinson index ($\varepsilon = 0.5$)	0.395	0.007	0.399	0.006
Mean log deviation – Generalised Entropy ($\alpha = 0$)	1.232	0.020	1.250	0.018
Theil index – Generalised Entropy ($\alpha = 1$)	0.931	0.037	0.942	0.032
Generalised Entropy ($\alpha = 2$)	5.186	1.276	4.411	0.655
Gini coefficient	0.680	0.006	0.685	0.005
Pietra index	0.495	0.004	0.501	0.004
Share indicators				
Top 5% share	37.2	1.0	37.8	0.8
Top 10% share	50.5	0.8	51.2	0.7
Quantile ratio indicators				
P90/P50 ratio	4.7	0.1	4.8	0.1
P80/P20 ratio	40.1	2.0	41.0	1.8

Sources: HFCS and authors' calculations.
Notes: Atkinson(0.5) computed for non-negative values of wealth only. Atkinson(1), GE(0) and Theil index computed only on positive values of wealth, following the conditions described in Table 1 and Table 2. Standard errors computed taking into account the multiple imputation and bootstrap weights.

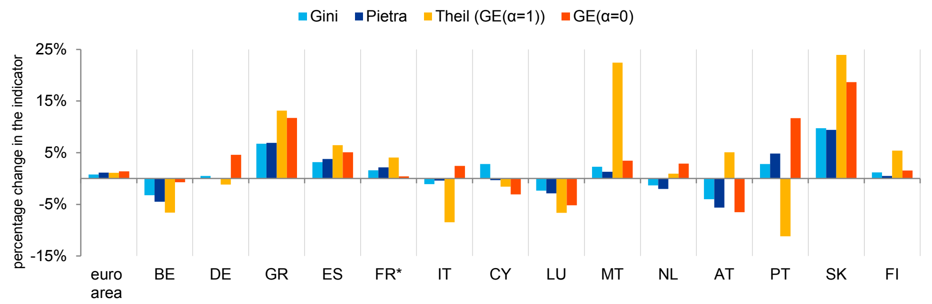
5.3 Evolution of wealth inequality in the euro area between 2010 and 2014: investigation into the reported modest increase

In the results of the second wave of the HFCS in 2016 the Gini coefficient and other indicators pointed to a modest increase in wealth inequality in the euro area in the period 2010-2014.[15] These results should be complemented with the changes taking place in each of the euro area countries and using different measures of inequality. The changes in the Gini, Pietra, Theil and Generalised Entropy (with parameter $\alpha = 0$) indicators in selected countries are given in Chart 2 (with the confidence intervals shown in Chart 3 and in Appendix section entitled "Complementing charts").

[15] As the only exception, the Generalised Entropy ($\alpha = 2$) showed a decrease in inequality (see Table 3).

Chart 2

Percentage change in wealth inequality in euro area countries between 2010 and 2014



Sources: HFCS and authors' calculations.
Notes: FR*: results for France corrected for the lack of information on the value of vehicles in the first wave of the survey. Observations with non-positive values have been dropped from the calculations of the Theil and GE(0) indicators. Changes between 2010 and 2014 for most countries (see exceptions in HFCS (2016)).

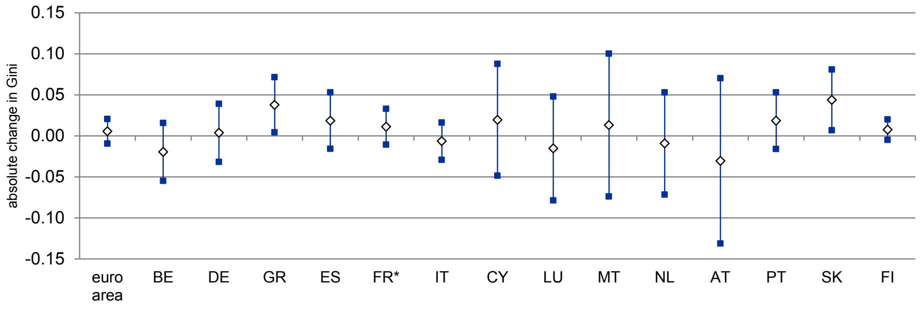
Results show that the changes in the indicators in the different countries are, in general, more pronounced than the aggregated euro area changes. More interestingly, the changes occur in different directions: the selected indicators point towards a decrease in wealth inequality in Belgium and Luxembourg and towards an increase in Greece, Spain, France, Malta, Slovakia and Finland. As for the other countries, the results are mixed, reflecting important differences in the nature of the change in the distribution. The Generalised Entropy index with parameter $\alpha = 0$ is more sensitive to transfers at the bottom of the distribution (when compared to the same indicator with a larger parameter α) and, as an example, points to a decrease in inequality in Austria and an increase in Portugal, whereas the Theil index (relatively more sensitive to transfers at the top of the distribution) gives the opposite results: an increase in inequality in Austria and decrease in inequality in Portugal.

These results should, however, be interpreted with care as the estimation of standard errors of the measures implies that the changes are only statistically significant at the 5% level in a few cases.

Chart 3 shows that the changes in the Gini coefficient are only statistically significant in Greece and Slovakia. The same holds for Atkinson's indicator with parameter $\varepsilon = 1$ (shown in Chart A.1 in the Appendix), where the increase in Portugal is also statistically significant. None of the changes as measured by Theil's index are statistically significant (shown in Chart A.2 in the Appendix).

Chart 3

Absolute change in Gini with 95% confidence interval

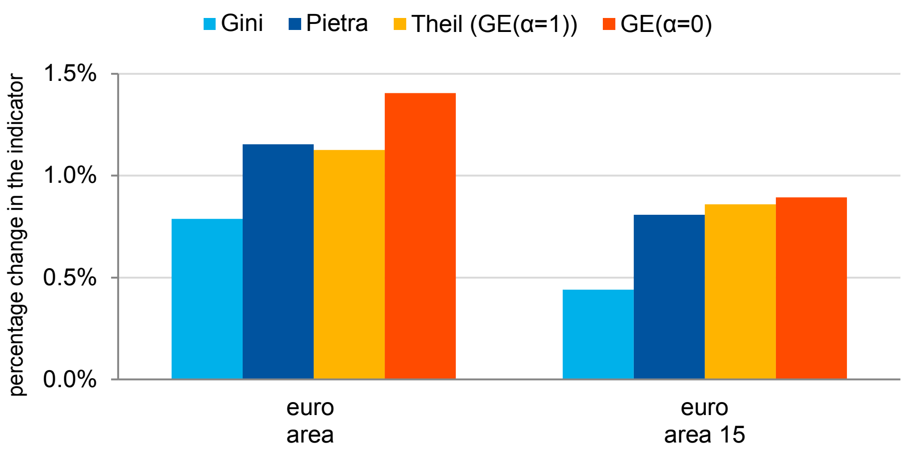


Sources: HFCS and authors' calculations.
Notes: FR*: results for France corrected for the lack of information on the value of vehicles in the first wave of the survey. Changes between 2010 and 2014 for most countries (see exceptions in HFCS (2016)).

Another aspect to consider is the composition of the euro area used for computing the indicators. Indeed, the euro area results of the HFCS are in general computed considering the euro area with a changing composition, i.e. it is defined as the 15 participating countries in the first wave of the survey and the 18 participating countries in the second wave. The use of a fixed composition of the 15 countries participating in both waves of the survey gives slightly different results, with even smaller changes in the indicators, when compared with the euro area with a changing composition (see Chart 4).

Chart 4

Percentage change in wealth inequality: different euro area compositions



Sources: HFCS and authors' calculations.
Note: Changes between 2010 and 2014 for most countries (see exceptions in HFCS (2016)). See also footnote to Chart 2.

5.4 Differences in inequality trends by indicator: contributions to total change from Pigou-Dalton transfers

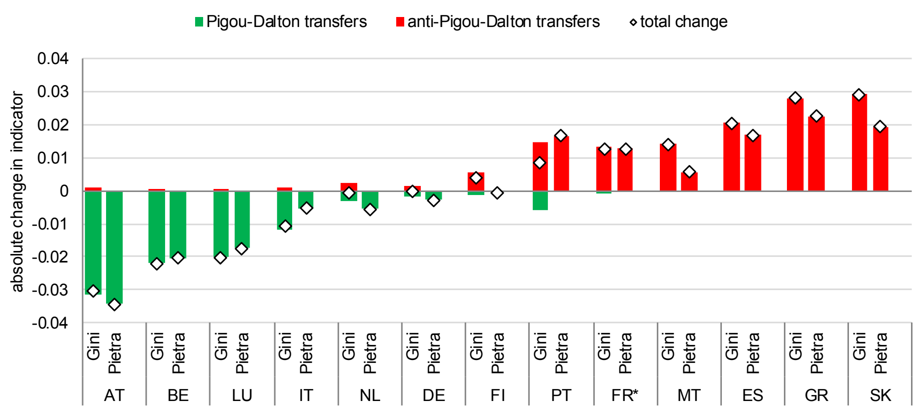
The results in the previous subsection show that, in some countries, the inequality indicators point in opposite directions as regards the changes in the inequality of the wealth distribution between the first two waves of the survey.

To see how the choice of specific indicator influences the result, we express, for each country, the difference between the distributions of the first and second waves of the HFCS as a sequence of Pigou-Dalton and anti-Pigou-Dalton transfers, applying the method introduced in Section 3.2. The absolute change in the inequality indicators is expressed as a sum of the contributions of Pigou-Dalton and anti-Pigou-Dalton transfers to the overall indicator.

Chart 5 illustrates how the same Pigou-Dalton and anti-Pigou-Dalton transfers are accounted for differently by the inequality indicators.[16] In Greece there are only anti-Pigou-Dalton transfers, that is, there was an increase in inequality according to all the indicators. In all the other countries, both Pigou-Dalton and anti-Pigou-Dalton transfers occur, even if in most cases the contribution to the change in the Gini indicator is barely measurable from one of the two transfer types. The exceptions are the Netherlands, Germany, Finland, and Portugal. The chart also shows how the Pietra index remains unchanged when some of the Pigou-Dalton or anti-Pigou-Dalton transfers are performed. More interestingly, for most countries, the total contribution to the Pietra from one of the two transfer types is zero, meaning that these changes occur on the same side of the distribution relative to the mean. Using the Generalised Entropy indicators to account for the same transfers gives yet another picture, shown in Chart 6 below.

Chart 5

Contributions of Pigou-Dalton transfers to total change in inequality – Gini and Pietra indicators

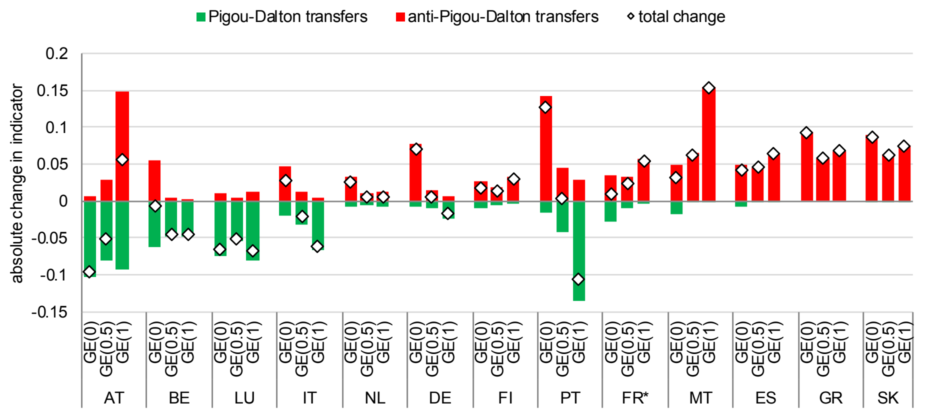


Sources: HFCS and authors' calculations.
Notes: FR*: results for France corrected for the lack of information on the value of vehicles in the first wave of the survey. Changes between 2010 and 2014 for most countries (see exceptions in HFCS (2016)).

[16] Owing to the complexity of the calculation of the components, the confidence intervals for the decomposition into Pigou-Dalton and anti-Pigou-Dalton components have not been shown in these graphs.

Chart 6

Contributions of Pigou-Dalton transfers to total change in inequality – Generalised Entropy indicators



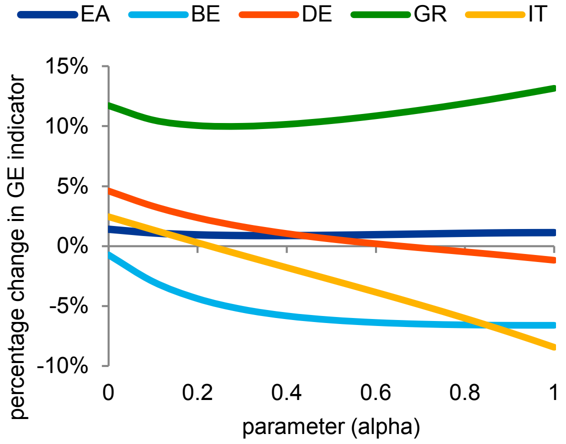
Sources: HFCS and authors' calculations.
Notes: FR*: results for France corrected for the lack of information on the value of vehicles in the first wave of the survey. Observations with non-positive values have been dropped from the calculations of the Generalised Entropy indicators. Changes between 2010 and 2014 for most countries (see exceptions in HFCS (2016)).

Unlike the contributions to the Gini coefficient, the contributions of Pigou-Dalton and anti-Pigou-Dalton transfers to the total change in the Generalised Entropy indicators are, in most cases, not negligible for both types of transfers. The type of transfer giving the biggest contribution can change depending on the parameter α chosen, resulting in a total change in the Generalised Entropy indicator, with it pointing in opposite directions in some cases, like in Austria, Italy, Germany or Portugal.

Chart 7 also shows how the choice of the parameter α when using the Generalised Entropy index affects the inequality trend for a selected number of countries. A higher value of α corresponds to a stronger importance of transfers at the top of the distribution, as discussed in Sections 2.2.2 and 3.2. In Germany and Italy, an increasing parameter α leads to a decrease in the change in the GE indicator, while for Greece the change is strongly positive in all cases. In Belgium, it is likely that a slightly negative value of the parameter would correspond to a positive change in the indicator.

Chart 7

Change in the Generalised Entropy indicator depending on the parameter α



Sources: HFCS and authors' calculations.
Notes: Observations with non-positive values have been dropped from the calculations. Changes between 2010 and 2014 for most countries (see exceptions in HFCS (2016)).

5.5 Sub-group dynamics: analysis using the difference between Lorenz curves

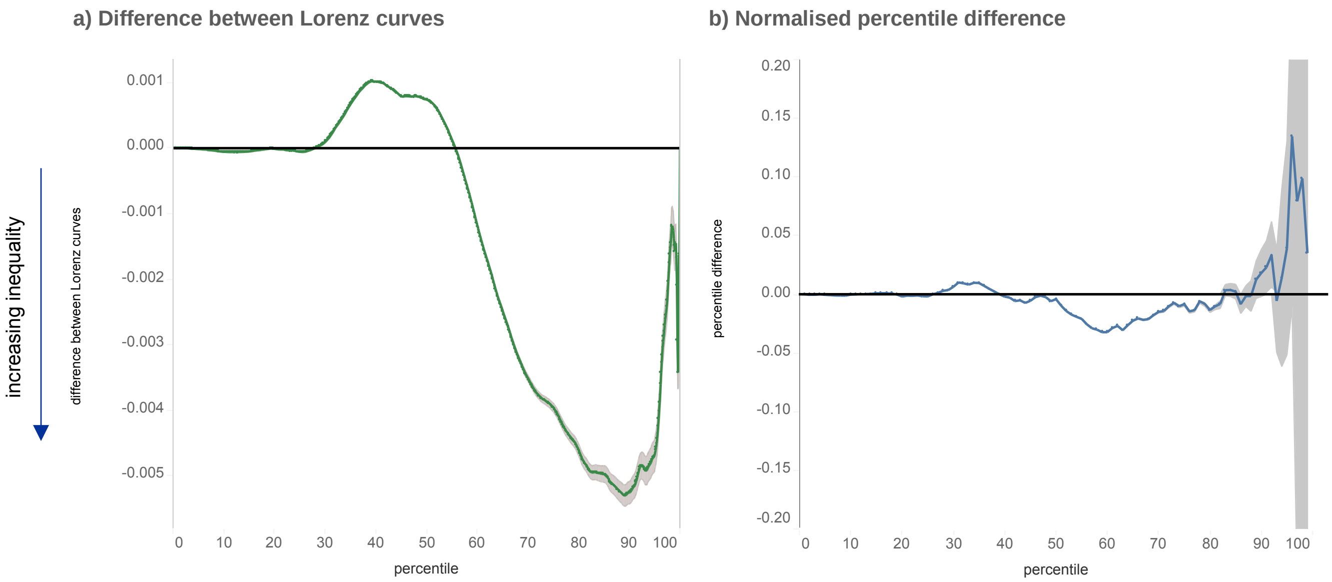
The change in wealth inequality can also be explained by the groups of households among which wealth transfers would need to be made to obtain the most recent distribution from the initial distribution. This analysis considers each household's wealth share. As described in Section 4.4, the intersections of the Lorenz curves of the two distributions being compared divide the population into sub-groups such that only one kind of wealth transfer (either Pigou-Dalton or anti-Pigou-Dalton) occurs within each of the sub-groups.

At the euro area level, the Lorenz curves of the two waves of the survey are relatively close to each other, intersecting close to the percentiles P30 and P55 and dividing the population into three sub-groups (Chart 8a). This means that it is possible to move from one distribution to another by redistributing wealth within these sub-groups, namely among the 30% poorest households, then among those between the 30% and 55% poorest, and finally among the 45% richest households. In each of these sub-groups, the share of total wealth has stayed the same across waves, and only the distribution within the sub-group may have changed. The difference between the Lorenz curves is positive for the "middle" sub-group, meaning that the wealth distribution within this sub-group has become more equal between the two waves of the survey. Within the two remaining sub-groups, that is, those comprising the poorest 30% and the richest 45%, inequality has increased. As the difference in the area between Lorenz curves is negative and directly related to the changes in the Gini, the charts allow us to conclude that most of the changes in the Gini between 2010 and 2014 were driven by changes in the top 45% of the distribution.

Chart 8b shows the relative wealth increase or decrease between the two waves of the survey for each percentile group.[17] In particular, we can see that the transfers occurring within the sub-group containing the poorest 30% to 55% of households are Pigou-Dalton transfers taking place from households in the percentiles P40 to P55 to those in the percentiles P30 to P40, therefore decreasing inequality in this population sub-group. Chart 8b further highlights the importance of the top of the distribution in the analysis of wealth inequality, as the changes at the top have a strong impact on the overall analysis.

Chart 8

Difference between Lorenz curves and normalised percentile difference: euro area



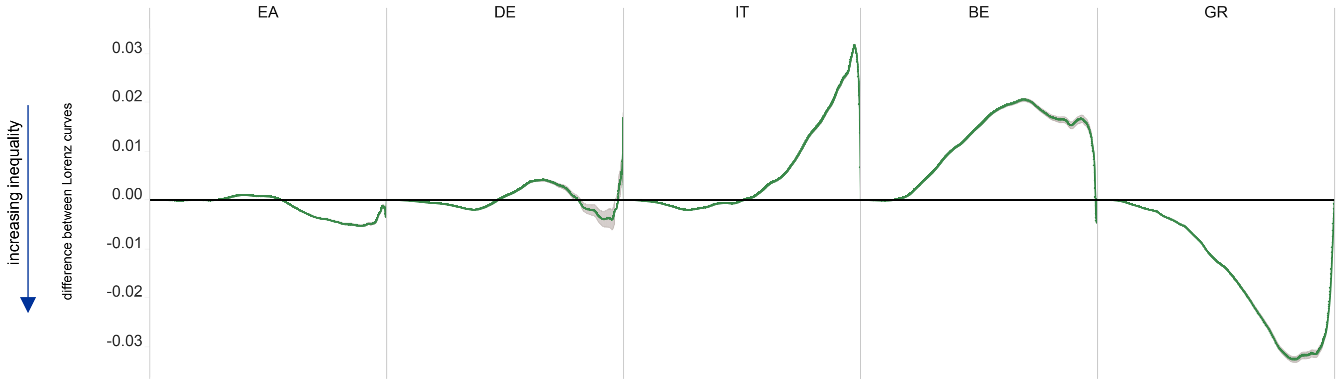
Sources: HFCS and authors' calculations.
Notes: Observations with non-positive values have been dropped from the calculations. Changes between 2010 and 2014 for most countries (see exceptions in HFCS (2016)). 95% confidence intervals are shown in light grey (Charts 8a and 8b) and truncated at [-0.20, 0.20] (Chart 8b).

The same analysis made at the country level again shows the differences across countries that are hidden in the aggregated euro area results. The differences between Lorenz curves are plotted in Chart 9 for a selection of countries.

[17] This quantile difference chart, inspired by the work of Kennickell (2009, 2011), normalises the wealth in the two periods to focus only on redistribution effects; in some sense the quantile difference chart is the first derivative of the difference between the Lorenz curves.

Chart 9

Difference between Lorenz curves of the 2010 and the 2014 wealth distributions – selected countries



Sources: HFCS and authors' calculations.
Notes: Observations with non-positive values have been dropped from the calculations. Changes between 2010 and 2014 for most countries (see exceptions in HFCS (2016)). 95% confidence intervals are shown in grey.

Different patterns arise: in Germany the population is divided into four sub-groups, with inequality increasing in the group made up of the poorest half of the population and in the group comprising households between percentiles P82 and P98. In the other two sub-groups, the first made up of households between P48 and P82 and the second of the 2% richest households, inequality has decreased.

The situation is different in Italy, with only two sub-groups arising: the poorest 51% households and the richest 49%. Between 2010 and 2014, wealth inequality increased among the households comprising the former sub-group and decreased among those in the second sub-group.

In Belgium the wealth distributions of the two waves of the survey are almost overlapping until the household at the 14th percentile. Inequality then decreased in the sub-group comprising the households between the 14th and 99th percentiles.

In Greece the difference between Lorenz curves is always negative as the curves do not intersect; the curve corresponding to the second wave of the survey always lies below that of the first wave. Therefore, according to Atkinson's comparison criterion and to any definition of wealth inequality consistent with the principle of transfers, wealth inequality has increased in Greece over the time period considered.

These results can be further compared to those of Section 5.4. Charts 6 and 7 show how, in Germany and Italy, inequality measured with Generalised Entropy indicators with the lower parameter α has tended to increase. For Generalised Entropy indicators, the lower the parameter, the greater the weight attached to transfers at the bottom of the distribution. Indeed, Chart 9 shows that, in Germany and Italy, inequality increased at the bottom of the distribution. Yet, when using values of the parameter equal to or greater than 0.5, these GE indicators show that, overall, inequality has decreased for these countries. On the contrary, with low values of the α parameter, GE indicators report an increase.

5.6 Contributions from wealth components

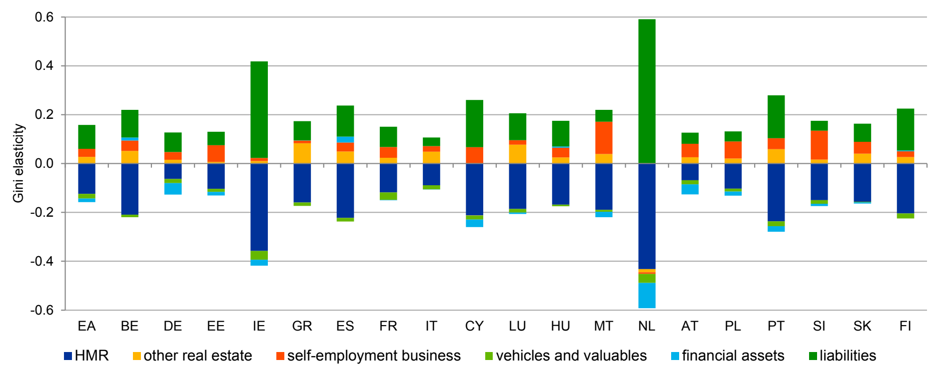
5.6.1 Marginal effects

Net wealth is measured in the HFCS as the difference between the household's total assets and liabilities. Assets can then be split into real assets – the value of the household main residence, other real estate, vehicles and valuables and self-employment businesses – and financial assets. Financial assets can be further decomposed into safe, risky and other financial assets[18]. Lindner (2015) presented a decomposition of wealth along these components. We conduct a similar analysis for the Gini and for the other main inequality indicators, starting with the second wave of the HFCS before considering the changes between the two waves in the following section.

The elasticities of the Gini coefficient with respect to the mean for each component of total wealth were introduced in Section 4.5.2 and are presented in Chart 10. By construction, the sum of the elasticities is null: an increase of all components of wealth by 1% leaves inequality unchanged overall. Since the total effect is equal to the sum of the effects of the individual components, for the total to be zero, some effects have to have opposite signs, meaning there will always be equalising and disequalising components under this decomposition of marginal effects.

Chart 10

Elasticity of the Gini coefficient, second wave – wealth components



Sources: HFCS and authors' calculations.

Across all countries, the household main residence and vehicles and valuables have an equalising effect (i.e. a proportional increase in the value of HMR for all households decreases the Gini coefficient, other things being equal) whereas liabilities and self-employment businesses have a disequalising effect in all countries. It might appear surprising that an increase in the value of an asset that is only held by a fraction of the population decreases inequality. However, this is explained by two

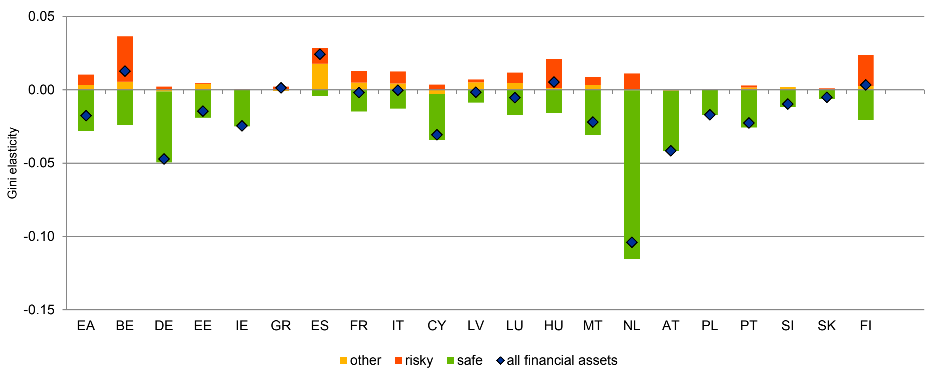
[18] HFCS variables, with reference to the derived variables in HFCS (2016): Safe financial assets = deposits (DA2101) + voluntary pension/whole life insurance (DA2109); Risky financial assets = mutual funds (DA2102) + bonds (DA2103) + shares, publicly traded (DA2105); Other financial assets = value of non self-employment private business (DA204) + managed accounts (DA2106) + money owed to households (DA2107) + other assets (DA2108).

facts, namely that i) the ownership of both the HMR and vehicles and valuables is more widespread than for most other components; and ii) there is less inequality in the distribution of these two types of assets than, for example, self-employment businesses, of which the ownership is concentrated and the range of values of businesses is much wider, especially at the top of the wealth distribution (see Chart 1). For most less-wealthy households, the HMR is the main or even the only asset while for wealthier households the HMR makes up a smaller fraction of assets. In countries where ownership of the HMR is less prevalent (Austria and Germany) the marginal effect of the HMR is both smaller and of comparable size to that of financial assets, while in other countries the marginal effect of financial assets is smaller. However, we should point out that this analysis on wealth inequality does not take into account the fact that housing is both an asset and a consumption item and that the increase in the value of HMR also increases the imputed consumption of housing services of households renting their main residence; nevertheless these considerations are not pursued here further.

Financial assets have an equalising effect in most countries but not in Belgium and Spain, for example, although the effect is moderate in these two countries. Other real estate has a disequalising effect in all countries except Cyprus, the Netherlands and Latvia. Chart 11, which presents the Gini elasticities of financial assets, shows that safe financial assets have equalising effects in all countries. Risky and other financial assets mostly have small, disequalising effects in most countries.

Chart 11

Elasticity of the Gini coefficient, second wave – financial assets

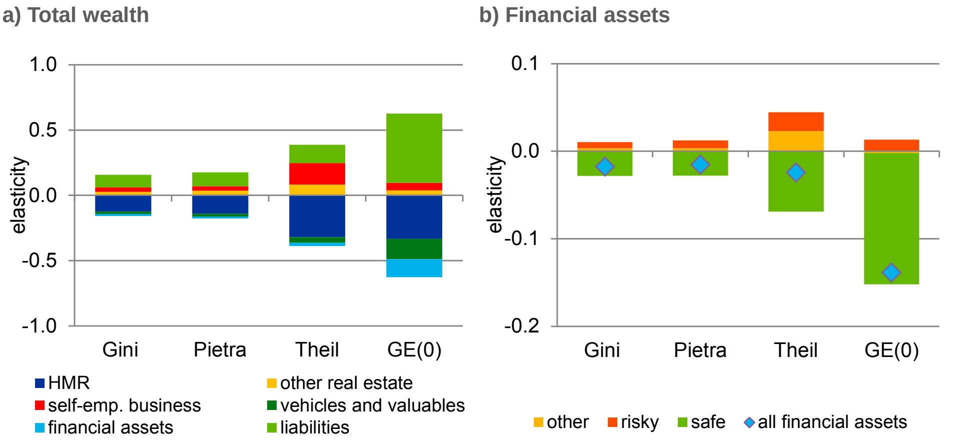


Sources: HFCS and authors' calculations.

The elasticities computed for other indicators (Charts 13 and 14) described in Sections 4.5.2 and 4.6 show the contributions of each of the components. Interestingly, all elasticities are of the same sign as the Gini elasticities, although the magnitude varies. The Pietra elasticities are very close to those of the Gini, while the Theil index GE(1) and the GE(0) index show more variation. The equalising role of financial assets and, in particular, safe financial assets is much stronger for the GE(0) measure.

Chart 12

Elasticity of other inequality indicators – total wealth



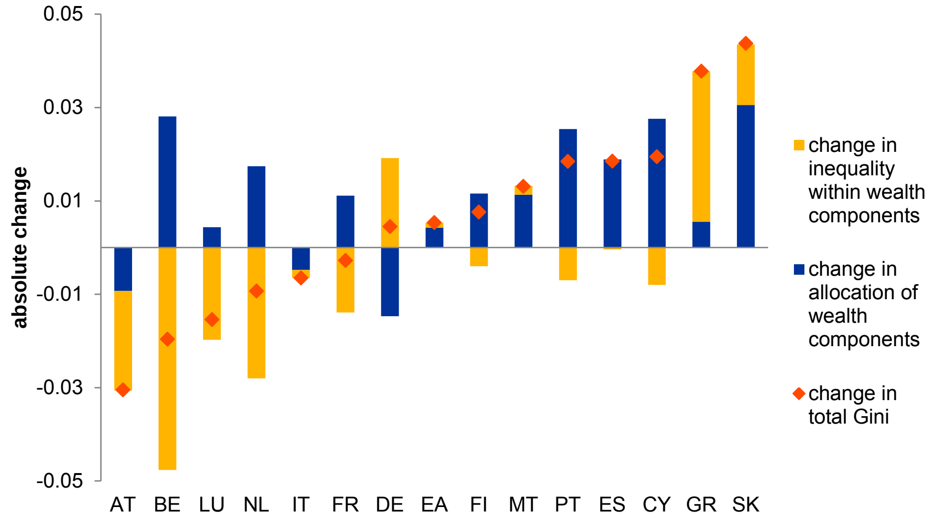
Sources: Second wave of the HFCS, euro area, and authors' calculations.
Note: Observations with non-positive values have been dropped from the calculations of the Theil and GE(0) indicators.

5.6.2 Decomposition of change in Gini

The change in the Gini coefficient between the two waves can be decomposed into two factors (as described in Section 4.5.3), taking as components the constituents of total wealth described in the previous section, namely household main residence, other real estate, vehicles and valuables, self-employment businesses and financial assets. One factor accounts for the change in inequality within each component ("change in intrinsic inequality") and the other accounts for a change in the allocation of the different components ("change in share"). Chart 13 shows how these two factors add up to the total change in inequality, as measured by the Gini coefficient for the euro area and for each country.

Chart 13

Absolute change in the Gini coefficient between waves and its decomposition



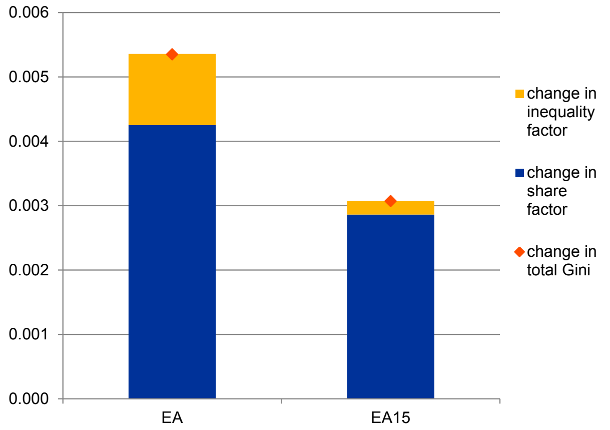
Sources: HFCS and authors' calculations.
Notes: Changes between 2010 and 2014 for most countries (see exceptions in HFCS (2016)). Countries are sorted by increasing change in total Gini. The components adding up to net wealth are household main residence, other real estate, vehicles and valuables, self-employment businesses and financial assets.

The "change in inequality" factor only contributed positively to the change in total Gini in Germany, Greece, Malta and Slovakia. In all the other countries where the Gini coefficient increased, the increase can only be explained by a change in the allocation of the wealth components. Furthermore, the change in the allocation of the wealth components mainly contributed to an increase in total inequality, with the only exceptions being Austria, Italy and Germany.

In the euro area, the "change in inequality" factor also contributed to the increase in inequality, albeit with a very small contribution when compared with the "change in allocation" factor. Moreover, this factor is even smaller when considering the euro area with unchanged decomposition between waves (Chart 14).

Chart 14

Change in Gini coefficient between waves and its decomposition for the euro area



Sources: HFCS and authors' calculations.
Notes: Changes between 2010 and 2014 for most countries (see exceptions in HFCS (2016)). The components adding up to net wealth are household main residence, other real estate, vehicles and valuables, self-employment businesses and financial assets.

6 Conclusion

A change in an abstract indicator usually hides the significant complexity of its interpretation and explanation. The modest increase in wealth inequality over the first two waves of the HFCS is a case in point for this observation.

This paper started as an attempt to shed some light in the correct measurement of wealth inequality and, more importantly, understand its determinants through appropriate sub-indicators. It took a very brief path through the rich literature on the axiomatic specification of inequality indicators, pointing out the relationship between the Gini coefficient, the Pietra index, top shares, and quantile ratios, with the last two indicators lacking some important characteristics desired in inequality measures. It also introduced Atkinson's measure, Theil's measure, and their family of Generalised Entropy indices. All these different measures are, however, sensitive to different parts of the distribution and can lead to different measures of changes – over time or over space.

Atkinson's comparison criterion is an important stepping stone to address this problem and makes it possible to state "absolute" comparisons over a large family of inequality measures verifying the principle of transfers (stated loosely, "transferring wealth down the distribution always decreases inequality"). We then built a novel decomposition of changes between two distributions with a succession of Pigou-Dalton transfers that were either all equalising or all disequalising within contiguous sub-groups of the population, ensuring uniqueness of the decomposition in two non-negative factors. Atkinson's comparison criterion translates in to one of these two factors being exactly zero; any deviation from this implies that there is an inequality measure that would show a change in the opposite direction.

The literature has also proposed a large number of decompositions of inequality measures or of their changes, some of which can be applied to several of the different measures studied in the paper, which we briefly introduced.

Using the results of the first two waves of the HFCS allowed us to achieve two objectives: to illustrate how the various indicators and their decompositions can be applied to real wealth and to return to the matter that motivated the paper, namely understanding the nature of the changes in wealth between 2010 and 2014. The small increase in wealth inequality at the euro area level, and the relative agreement between the different inequality measures, is not replicated in several of the countries. Our decomposition into Pigou-Dalton transfers shows that in most countries the change in the Gini coefficient can only be understood as a set of transfers of the same kind, either equalising or disequalising. In the few countries where this is not the case (Netherlands, Germany, Finland, Portugal, and to a lesser degree Italy and France), the different inequality measures do lead to changes of opposite sign. As seen with Atkinson's comparison criterion and the decomposition into Lorenz sub-groups, the increase in inequality in the euro area is almost entirely due to the increase in wealth inequality of the top 50% of the population, although results vary strongly by country.

The different decompositions show the elasticity of the different components of wealth, with the household's main residence and safe financial assets being an equalising component (i.e. a proportional increase in the value of the main residence or of the safe financial assets decreases inequality, all other things being equal). Other real estate, self-employment businesses, and liabilities are disequalising, as they are held only by a smaller fraction of the population, and correlated with total wealth. Looking at the changes between the two waves of the survey, the change in the allocation of the components of wealth has tended to increase inequality, while the change in the distribution of each component of wealth drives the overall change in inequality: countries with a decrease in within-wealth component inequality are also the countries with the strongest decrease in wealth inequality.

Summing up, the changes in wealth between 2010 and 2014 in the euro area can be described as a modest increase, with a sign that is robust to most inequality measures. At the country level, measurement precision is lower and points to ambiguous changes in several countries (Germany, France, Italy, Netherlands, Portugal, Finland) while in others the results are more clear cut.

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Appendix

Expressions for inequality indicators

Let y_i represent the wealth of household $i \in \{1, \dots, n\}$. The vector $y = (y_1, y_2, \dots, y_n)$ represents the wealth of all observed households. Let w_i represent the weight of household i.

In the case where y is such that $y_1 \le y_2 \le \dots \le y_n$, we define:

$X_i \coloneqq \frac{\sum_{j=1}^{i} w_j}{\sum_{j=1}^{n} w_j}$ as the cumulative weight and $Y_i \coloneqq \frac{\sum_{j=1}^{i} y_j w_j}{\sum_{j=1}^{n} y_j w_j}$ as the cumulative wealth.

The table below gives one possible expression for each indicator in two cases: 1) generic weight and 2) in the case where all the individuals have the same weight.

Table A.1
Weighted and unweighted formulas for the different inequality indicators

Indicator	Generic weight w_i	Constant weight $w_i = \frac{1}{n}\ \forall i$
Gini	$G = 1 - \sum_{i=0}^{n-1} (X_{i+1} - X_i)(Y_{i+1} + Y_i)$	$G = \frac{2\sum_{i=1}^{n} i\, y_i}{n \sum_{i=1}^{n} y_i} - \frac{n+1}{n}$
Generalised Entropy (α)	$\text{GE}(\alpha) = \frac{1}{\alpha(\alpha-1)} \sum_{i=1}^{n} \frac{w_i}{\sum_{i=1}^{n} w_i} \left(\left(\frac{y_i}{\bar{y}} \right)^{\alpha} - 1 \right)$	$\text{GE}(\alpha) = \frac{1}{\alpha(\alpha-1)} \frac{1}{n} \sum_{i=1}^{n} \left(\left(\frac{y_i}{\bar{y}} \right)^{\alpha} - 1 \right)$
Theil (GE(α = 1))	$\text{GE}(1) = \sum_{i=1}^{n} \frac{w_i}{\sum_{i=1}^{n} w_i} \frac{y_i}{\bar{y}} \ln\left(\frac{y_i}{\bar{y}} \right)$	$\text{GE}(1) = \frac{1}{n} \sum_{i=1}^{n} \frac{y_i}{\bar{y}} \ln\left(\frac{y_i}{\bar{y}} \right)$
Mean log deviation (GE(α = 0))	$\text{GE}(0) = -\sum_{i=1}^{n} \frac{w_i}{\sum_{i=1}^{n} w_i} \ln\left(\frac{y_i}{\bar{y}} \right)$	$\text{GE}(0) = -\frac{1}{n} \sum_{i=1}^{n} \ln\left(\frac{y_i}{\bar{y}} \right)$
Atkinson (ε ≠ 1)	$\text{Atk}(\varepsilon) = 1 - \frac{1}{\bar{y}} \left(\sum_{i=1}^{n} \frac{w_i}{\sum_{i=1}^{n} w_i} y_i^{1-\varepsilon} \right)^{\frac{1}{1-\varepsilon}}$	$\text{Atk}(\varepsilon) = 1 - \frac{1}{\bar{y}} \left(\frac{1}{n} \sum_{(i=1)}^{n} y_i^{1-\varepsilon} \right)^{\frac{1}{1-\varepsilon}}$
Atkinson (ε = 1)	$\text{Atk}(1) = 1 - \frac{1}{\bar{y}} \exp\left(\sum_{i=1}^{n} \frac{w_i}{\sum_{i=1}^{n} w_i} \ln y_i \right)$	$\text{Atk}(1) = 1 - \frac{1}{\bar{y}} \left(\prod_{i=1}^{n} y_i \right)^{\frac{1}{n}}$
Pietra	$P = \frac{1}{2} \sum_{i=1}^{n} \left| \frac{w_i}{\sum_{i=1}^{n} w_i} - \frac{y_i w_i}{\sum_{i=1}^{n} y_i w_i} \right|$	$P = \frac{1}{2n} \sum_{i=1}^{n} \frac{\lvert y_i - \bar{y} \rvert}{\bar{y}} = \frac{1}{n} \sum_{y_i > \bar{y}} \frac{y_i - \bar{y}}{\bar{y}}$

Decomposition of inequality changes by Pigou-Dalton transfers

This is the proof of the theorem presented in Section 3.2.2.

The algorithm can be written as follows:

Initial distribution $y^1 = (y_1^1, \dots, y_n^1)$

Final distribution $y^M = (y_1^M, \dots, y_n^M)$

Elements sorted ascendingly $y_i^1 \leq y_{i+1}^1$and $y_i^M \leq y_{i+1}^M$ $\forall i \in \{1, \dots, n\}$; equal means $\overline{y^1} = \overline{y^M}$

Let $c_t^y(y_k, y_j)$ denote the transfer of amount t performed on distribution y from individual y_k to individual y_j.

Let the distribution y^j resulting from step j of the algorithm be such that $y^j = (y_1^j, \dots, y_n^j)$, $\forall i \in \{1, \dots, n\}$ $y_i^j \leq y_{i+1}^j$ and $\overline{y^J} = \overline{y^1} = \overline{y^M}$.

Then step $j+1$ is obtained as follows:

d^j is the vector of n elements $d^j = (d_1^j, \dots, d_n^j)$ where $\forall i \in \{1, \dots, n\}$ $d_i^j = y_i^j - y_i^M$

Set $a_j = \min\{i \in \{1, \dots n\} \mid d_i^j < 0\}$

Set $b_j = \min\{i \in \{1, \dots n\} \mid d_i^j > 0\}$

Set $t_j = \min\left(\left|d_{a_j}^j\right|, \left|d_{b_j}^j\right|\right)$

y^{j+1} is obtained from y^j by transfer $c_{t_j}^{y^j}\left(y_{b_j}^j, y_{a_j}^j\right)$, i.e. $y_i^{j+1} = \begin{cases} y_i^j + t_j \; if \; i = a_j \\ y_i^j - t_j \; if \; i = b_j \\ y_i^j \text{ otherwise} \end{cases}$ and the elements are sorted subsequently.

The algorithm described above will, by construction, result in transfers that occur exclusively inside each of the groups defined by the intersections of the Lorenz curves of the initial and final distributions (LCG), as defined in Section 4.4. In each of these groups, the initial distribution can be transformed into the final distribution exclusively through either Pigou-Dalton or anti-Pigou-Dalton transfers among elements of the group. This procedure will result in a sequence of transfers that minimises the total change in the Gini coefficient and the magnitudes of the contributions of Pigou-Dalton and anti-Pigou-Dalton transfers to this change.

Let S represent the sequence of distributions that result from performing the algorithm. Let $PD(S)$ (respectively, $aPD(S)$) denote the set of all Pigou-Dalton (respectively, anti-Pigou-Dalton) transfers in S. PD and aPD are used below for a lighter reading.

We recall the expression for the total change in the Gini coefficient:

$$\text{Gini}(y^M) - \text{Gini}(y^1) = C_{PD}^{\text{Gini}} + C_{aPD}^{\text{Gini}}$$

We take the case of an LCG sub-group where only Pigou-Dalton transfers occur.

$$\min_S\{|C_{PD}^{Gini}|\} = \min_S\left\{\sum_{c_t^i \in PD} \left|\Delta\text{Gini}(c_t^i)\right|\right\} = \min_S\left\{\sum_{j<k} \frac{2t|j-k|}{n\sum_i y_i}\right\}$$

Minimising the expression above can be seen as the problem of choosing the specific pairs j, k and the corresponding transfer t. Furthermore, it can be written as a particular case of the Earth Mover's Distance (EMD) problem, of finding the flow f_{ij}

that minimises the EMD: $\sum_{i,j} f_{ij} m_{ij}$ where m_{ij} is the distance between elements i and j. In the case of only Pigou-Dalton transfers, $m_{ij} = i - j$ and $\forall i \leq j, f_{ij} = 0$.

We then have: $EMD = \sum_i \sum_{j<i} f_{ij}(i-j) = \sum_i i \sum_{j<i} f_{ij} - \sum_i \sum_{j<i} j\, f_{ij} = \sum_i i\, D_i - \sum_j j\, \sum_{i>j} f_{ij} = \sum_i i\, D_i - \sum_j j\, R_j = \sum_i i\,(D_i - R_i)$ where $D_i = \sum_{j<i} f_{ij}$ is the total wealth given by individual i and $R_j = \sum_{i>j} f_{ij}$ is the total wealth received by individual j. This implies that the EMD does not depend on the exact sequence of flows f_{ij} but rather only on the total wealth to be received or given.

The EMD is thus independent from the specific transfers, as long as they are all Pigou-Dalton (symmetrically, this holds when all transfers inside the LCG are anti-Pigou-Dalton). This means that, inside an LCG, the choice of specific Pigou-Dalton transfers is irrelevant to the group's contribution to the total change in the Gini, and the contribution of the Pigou-Dalton transfers is thus minimal.

Expressions for decompositions of inequality indicators

Population sub-groups: let X represent the complete population formed by n households that can be partitioned into K sub-groups of n_k households each, such that $X = \bigcup_{k=1}^{K} X_k$ and $n = \sum_{k=1}^{K} n_k$. $\overline{y_k}$ is the sample average of sub-group X_k.

Variable components: the variable net wealth can be expressed as a sum of R components such that $y_i = \sum_{r=1}^{R} y_{ir}$ where y_{ir} is the value of component r for household i. $\overline{y_r}$ is the sample average of component r. F_r denotes the theoretical distribution of the wealth component r.

Table A.2
Decomposition by population sub-groups – Generalised Entropy

Indicator	Within	Between
$GE_\alpha(X)$ $=$ within + between	$\sum_{k=1}^{K} \frac{n_k}{n}\left(\frac{\bar{y}_k}{\bar{y}}\right)^\alpha \mathrm{GE}_\alpha(X_k)$ where $\mathrm{GE}_\alpha(X_k) = \frac{1}{\alpha(\alpha-1)} \frac{1}{n_k} \sum_{i=1}^{n_k}\left(\left(\frac{y_{ik}}{\bar{y}_k}\right)^\alpha - 1\right)$	$\mathrm{GE}_\alpha(\cup_{k=1}^{K} X_k^*)$ where $\mathrm{GE}_\alpha(\cup_{k=1}^{K} X_k^*) = \frac{1}{\alpha(\alpha-1)} \frac{1}{K} \sum_{k=1}^{K}\left(\left(\frac{\bar{y}_k}{\bar{y}}\right)^\alpha - 1\right)$

Table A.3
Decomposition by non-overlapping population sub-groups – Gini coefficient

Indicator	Within (G_w)	Between (G_{nb})
Gini = within + between = $G_w + G_{nb}$	$\sum_{k=1}^{K} G_k \frac{n_k}{n} \frac{n_k \overline{y_k}}{n\bar{y}}$ where $G_k = \frac{\sum_{i=1}^{n_k}\sum_{j=1}^{n_k} \lvert y_{ik} - y_{jk}\rvert}{2 n_k^2 \overline{y_k}}$	$\sum_{k=1}^{K}\sum_{h=1}^{k-1} G_{kh}\left(\frac{n_k}{n}\frac{n_h \overline{y_h}}{n\bar{y}} + \frac{n_h}{n}\frac{n_k \overline{y_k}}{n\bar{y}}\right)$ where $G_{kh} = \frac{\sum_{i=1}^{n_k}\sum_{j=1}^{n_h} \lvert y_{ik} - y_{jh}\rvert}{n_k n_h(\overline{y_k} + \overline{y_h})} = \frac{\overline{y_k} - \overline{y_h}}{\overline{y_k} + \overline{y_h}}$

Note: Case of non-overlapping sub-groups: $k, h \in \{1, \ldots, K\}$ such that $\min_{y_i}(X_k) > \max_{y_i}(X_h)$.

Table A.4

Alternative decomposition by population sub-groups – Gini coefficient

Indicator	Contribution of sub-group k
Gini $= \sum_{k=1}^{K} T_k$	$T_k = \frac{n_k}{n}\frac{\overline{y_k}}{\bar{y}} C_k$

C_k is the concentration coefficient of the sub-group vector $\underline{x_k}$. The elements of the sub-group are $x_{ik} = y_i \mathbb{1}_{\{i \in x_k\}}$ and the concentration coefficient is computed as the Gini coefficient of the curve that plots the cumulative proportions of vector $\underline{x_k}$ against the cumulative proportions of the total population ordered ascendingly according to their wealth.

Table A.5

Decomposition by variable components – Gini coefficient

Indicator	Gini correlation	Gini of component r	Share of component r
Gini $= \sum_{r=1}^{R} P_r G_r S_r = \sum_{r=1}^{R} C_r S_r$	$P_r = \frac{\text{cov}(y_r, F)}{\text{cov}(y_r, F_r)}$	$G_r = \frac{2\text{cov}(y_r, F_r)}{\bar{y}_r}$	$S_r = \frac{\bar{y}_r}{\bar{y}}$
	Concentration coefficient or pseudo-Gini: $C_r = P_r G_r$		

C_r is the concentration coefficient of the wealth component r. It is computed as the Gini coefficient of the curve that plots the cumulative proportions of wealth component r against the cumulative proportions of the total population ordered ascendingly according to their total wealth.

Table A.6

Marginal effects of variable components

Indicator	Elasticity with respect to the mean	Pseudo-indicator
Gini	$\eta_r(G) = \frac{S_r(C_r - G)}{G}$	$C_r = \frac{2\text{cov}(y_r, F)}{\bar{y}_r}$
Generalised Entropy	$\eta_r(\text{GE}(\alpha)) = \frac{\alpha S_r(\text{GE}(\alpha)'_r - \text{GE}(\alpha))}{\text{GE}(\alpha)}$	$\text{GE}(\alpha)'_r = \frac{1}{\alpha(\alpha-1)}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{y_{ir}}{\bar{y}_r}\left(\frac{y_i}{\bar{y}}\right)^{\alpha-1} - 1\right)$
Theil (GE($\alpha = 1$))	$\eta_r(\text{GE}(1)) = \frac{S_r(\text{GE}(1)'_r - \text{GE}(1))}{\text{GE}(1)}$	$\text{GE}(1)'_r = \frac{1}{n}\sum_{i=1}^{n}\frac{y_{ir}}{\bar{y}_r}\ln\left(\frac{y_i}{\bar{y}}\right)$
Mean log deviation (GE($\alpha = 0$))	$\eta_r(\text{GE}(0)) = \frac{(S_r - \bar{S}_r)}{\text{GE}(0)}$	N/A
Pietra	$\eta_r(P) = \frac{S_r(P'_r - P)}{P}$	$P'_r = \frac{1}{n}\sum_{i, y_i > \bar{y}}\frac{y_{ir} - \bar{y}_r}{\bar{y}_r}$

Table A.7

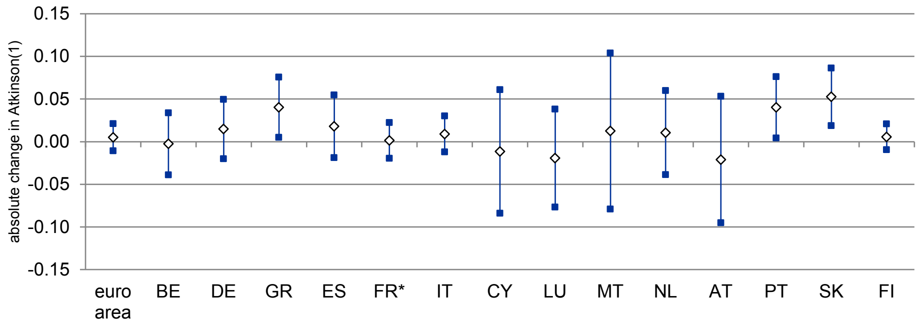
Decomposition of changes in time by variable components

Indicator	Change in share	Change in inequality
ΔG_t = change in share + change in inequality	$\sum_r \left(\frac{C_{r,t} + C_{r,t-1}}{2} - \frac{(G_{r,t} + G_{r,t-1})}{2}\right) \Delta S_{r,t}$	$\sum_r \frac{S_{r,t} + S_{r,t-1}}{2} \Delta C_{r,t}$
ΔG_t = change in share + change in inequality	$\sum_r (C_r - G)\,\dot{S}_r$	$\sum_r S_r \dot{C}_r$

Complementing charts

Chart A.1

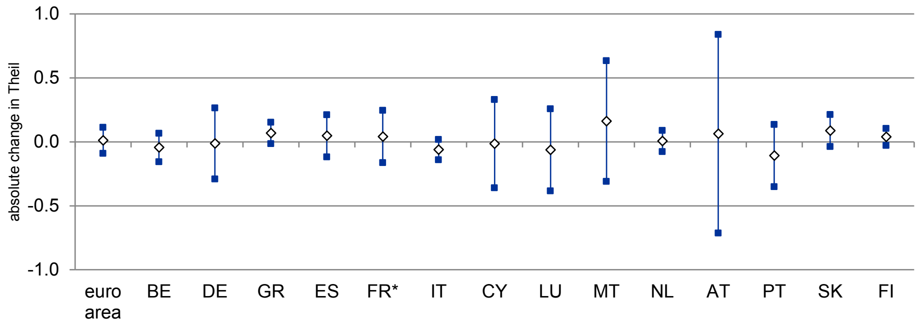
Absolute change in Atkinson(1) with 95% confidence interval



Sources: HFCS and authors' calculations.

Chart A.2

Absolute change in Theil with 95% confidence interval



Sources: HFCS and authors' calculations.

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